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ADVANCED COMMUNICATION THEORY TECHNIQUES

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Purdue University
Lafayette, Indiana

Authors: J. C. Hancock, Principal Investigator
D. G. Lainiotis
J. C. Lindenlaub
R. G. Marquart
H. Schwarzlander

NOTICES

FOREWORD

This report describes the studies undertaken on Air Force Contract AF 33(657)-7610 under Task No. 433502 of Project No. 4335 at the Communication Sciences Laboratory of Purdue University. The work was carried out under the direction of the Electromagnetic Warfare and Communications Laboratory, Aeronautical Systems Division, Wright-Patterson Air Force Base, Ohio.

The principal investigator wishes to acknowledge the suggestions of and discussions with the project engineer, Mr. B. W. Russell of ASD, as well as his associates.

The contributions of Mr. D. Weiner and Mr. D. J. Kostas to Chapters II and III respectively are also gratefully acknowledged.

ABSTRACT

Under this contract a number of topics have been studied and analyzed in detail in order to bring together and somewhat extend the concepts of communication theory as they apply to some current problems in digital communication systems.

Radio wave channels are characterized by a model which accounts for both multiplicative and additive disturbances. A large amount of experimental data pertaining to radio disturbances is evaluated and correlated. The importance of the Rayleigh fading channel is emphasized and previous work is extended to determine the capacity and efficiency of the Rayleigh channel.

Detection theory concepts have been extended to treat the problem of signal detection in the presence of statistically unknown additive disturbances. Several detectors based on non-parametric statistical techniques are treated in detail. These detectors are compared to the conventional likelihood detectors. Design procedures are formulated.

Signal design techniques are used to optimize transmitted waveforms and the improvement in system performance is determined. The criterion used in this analysis is the minimization of intersymbol influence and the minimization of transmitter power for a fixed probability of received errors.

The tradeoffs available between transmitter power and coding complexity are thoroughly investigated for the binary symmetric channel. Results are obtained for both Hamming and Bose-Chandhuri codes.

Recommendations for further work in promising areas are made. The need to supplement theoretical work with experimental work is pointed out.

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LIST OF IMPORTANT SYMBOLS

		Chapter
A	multiplicative channel gain	III
$G_n(f)$	noise power spectral density	"
$G_s(f)$	signal power spectral density	"
N_o	white noise power spectral density	"
N	noise power	"
P	signal power	"
W	information bandwidth	"
s(t)	sample function from a stationary random process	"
n(t)	sample function of a Gaussian noise process	"
H	rate of received information	"
C	channel capacity	"
I(S/X)	average mutual information	"
β	efficiency factor	"
E	energy	"
{N(t)}	noise random process	IV
N(t)	sample function from the noise random process {N(t)}	"
N'(t)	sample function from the noise random process {N(t)}	"
s(t)	signal function	"
Y(t)	input to the detector; sample function from {Y(t)}	"
A	amplitude of the signal	"
P(y)	probability distribution function of the random variable Y under no signal conditions	"
z	signal-to-noise ratio	"
$\bar{z}$	average signal-to-noise ratio	"

$p(y)$	probability density function of Y under no signal conditions	IV
$P_z(y)$	probability distribution function of Y under signal conditions $(z \neq 0)$	"
$p_z(y)$	probability density function of Y under signal conditions	"
y_i	samples from the random process $\{Y\}$	"
$E_o[U_{mn}]$	mean of the random variable U_{mn} under no signal conditions	"
$E_z[U_{mn}]$	mean of the random variable U_{mn} under signal conditions	"
$\sigma_o[U_{mn}]$	standard deviation of U_{mn} under no signal conditions	"
$\sigma_z[U_{mn}]$	standard deviation of U_{mn} under signal conditions	"
K	a constant	"
n	number of samples from $\{Y(t)\}$	"
m	number of samples from $N'(t)$	"
α	probability of false alarm	"
β	probability of false dismissal	"
U_{mn}	test statistic	"
$E_{U^*,U}$	asymptotic relative information efficiency of detector U* with respect to the detector U	"
U_α	threshold value; a value of the test statistic U resulting in a false alarm probability α	"
L_n	likelihood ratio	"
$\prod_{i=1}^{N}$	product of N terms	"
$\sum_{i=1}^{M}$	summation of M terms	"
$E(U)$	efficacy of the test statistic U	"

$\phi_o(y)$	probability density of the Gaussian random variable y under no signal conditions	IV
N	noise standard deviation	"
$\phi_z(y)$	probability density of the Gaussian random variable Y under signals conditions	"
t_n	optimum test statistic for the D-C detection problem	"
t_n'	optimum test statistic for the noncoherent detection problem	"
V_{mn}	the Mann-Whitney test statistic	"
P_e	the sum of the false alarm and false dismissal probabilities	"
Kmn	the Kolmogorov-Smirnov test statistic	"
$T_n(y)$	empirical distribution function of the sample $y_1 \dots y_n$	"
$S_m(y)$	empirical distribution function of the sample $y_1 \dots y_m$	"
K_α	a threshold value of Kmn resulting in false alarm probability α	"
R_{mn}	a rank test statistic	"
R^*_{mn}	the rank statistic for the D-C detection problem	"
T_{mn}	the rank statistic for the noncoherent detection problem	"
H_o'	hypothesis of signal being absent	"
H_1'	hypothesis of signal being present	"
a	pulse duration	V
a_i	constants	"
b	time duration	"
d	pulse duration	"
$e_i(t)$	input signal (into channel)	"
$e_o(t)$	output signal (from channel)	"
$e_1(t)$	a pulse waveform	"

h_i	coefficients	V
$h_{-1}(t)$	unit step response of network	"
j	$\sqrt{-1}$	"
k_i	coefficients	"
s	transform variable	"
$u(t)$	unit step function	"
C	capacitance	"
$D(s)$	polynomial in s	"
D_η	denominator of an expression	"
$E_i(s)$	Laplace transform of $e_i(t)$	"
$G_a(s)$	a certain class of entire functioning	"
$H(s)$	network transfer function	"
L	inductance	"
$N(s)$	polynomial in s	"
N_η	numerator of an expression	"
$P_i(s)$	polynomial in s	"
R	resistance	"
α	inverse time constant	"
γ	inverse time constant	"
γ_j	poles of a transfer function	"
δ	variation	"
η_p	pulse transmission efficiency	"
$\hat{\eta}_p$	optimum pulse transmission efficiency	"
λ	characteristic value	"
A_i	number of received error patterns, weight=j, for which the corrected word contains a given specific binit in error; independent of the binit chosen	VI

d_j	number by which reference is made to a specific binit in the code word	VI
E	energy per binit	"
e_j	number of the j^{th} binit in error in the received word	"
(e_j)	set of all binits in error in the received word; $(e_1, e_2, \ldots, e_i)$, when the word contains i errors	"
$e_j'(e_j')$	as for e_j, (e_j), but at the decoder output	"
k	number of information binits in a code word	"
L_i	number of error patterns (e_j) of weight i in a Hamming code word for which each of the corresponding (e_j') have weight i + 1	"
M_i	as for L_i, for which the weight of each (e_j') is i-1	"
N_i	as for L_i, for which the weight of each (e_j') is i	"
$N_{i\alpha}$	as for L_i, for which $d_\alpha \epsilon (e_j')$, where d_α is an information binit	"
$N_{i\alpha}'$	as for $N_{i\alpha}$, with the added condition that (e_j) is such that y=1 (see "y" below)	"
n, N	length of a Hamming SEC code word; $= 2^m - 1$, m = positive integer. "N" is used on figures; "n" is used in text.	"
n'	length of a Hamming SEC/DED code word; = n + 1	"
P_e	channel binit error probability	"
P_e'	decoder output binit error probability	"
y	indicator, Hamming SEC/DED codes; = 1 if the received word is retained, = 0 if the received word is discarded	"
z	indicator, Hamming SEC/DED codes; = 1 if the overall check binit is received in error; = 0 if the overall check binit is received correctly	"
ϵ, ϵ_i	binary sequences; possible received words	"
ω, ω_i	code words	"
$\bullet$	vector, modulo 2, addition	"

CHAPTER I

INTRODUCTION

A study of advanced Communication Theory Techniques was undertaken by the Communication Sciences Laboratory of Purdue University for the Aeronautical Systems Division, Wright Patterson Air Force Base during February 1962. The purpose of this program was to help unify present diversified aspects of statistical communication theory, stressing the interrelation which exists between information, decision and coding theories.

The major emphasis of this research is placed on the connecting of a number of theories to stress the roles which they play in determining the performance of a communication system. Although the major portion of this study was originally to be a collecting, simplifying, and integration of previous studies into a gross framework, it soon became apparent that considerable extensions were needed in a number of areas before this could be accomplished. Four primary areas of investigation were chosen for further study. These include: a) a discussion of channels, their characteristics and capacities, b) the use of non-likelihood detection to combat non-Gaussian noise sources, c) the application of signal design techniques to channels which have memory, and d) the trade-off in system parameters in a coded system. This report contains the results of studies made in the above areas.

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The principal problems and results derived from this study are summarized in this first chapter. The detailed discussion is presented in the remaining chapters of the report.

1.1 Channel Characterization

The characterization of radio wave channels is treated in detail in Chapter II. A simple model, useful in analysis, is presented which accounts for degradation in the received signal in terms of both multiplicative and additive disturbances. Additive and multiplicative disturbances commonly encountered in typical channels are discussed. The importance and applicability of the Rayleigh fading channel is pointed out. The chapter brings together and correlates a great deal of experimental data and results that were previously only to be found scattered throughout the technical literature.

1.2 Capacity of the Rayleigh Fading Channel

In Chapter III the capacity of the Rayleigh fading channel is derived. The results are compared with the capacity of the unity gain channel for different received signal-to-noise power ratios. In order to compare the Rayleigh channel to other channels, the efficiency factor β (defined as the required received energy per information bit received in the presence of a given Gaussian disturbance) is also evaluated.

1.3 Non-parametric Detection

The problem of detection of a signal in noise of known statistical properties has been investigated thoroughly in the past. However, these methods are completely inapplicable and inappropriate whenever these

noise statistics are unknown. In Chapter IV a detection criterion based on the methods of non-parametric statistics is utilized that permits the design of detectors on the basis of much less a-priori information. Several detectors based on this detection criterion are investigated and their properties obtained. A comparison between the optimum (likelihood) detectors and these new (non-likelihood) detectors is made on the basis of information efficiency. Also, a practical design procedure is formulated for the design of these new (non-likelihood) detectors.

1.4 Optimization of Signaling Waveforms

In Chapter V the application of Signal Design to digital communications is considered. This essentially involves two basic questions: (1) how can the transmitted waveforms be optimized; and (2) how much improvement in system performance may be achieved in this manner. It is pointed out that many factors combine to determine the best signal to be transmitted in any particular situation, among these being the characteristics of the channel and the criterion of performance.

In the work performed thus far, a dispersive channel with additive Gaussian noise is considered. Radio transmissions through - or reflected or scattered by - the ionosphere are examples of such channels, where the dispersive nature arises from the existence of some continuous range of path lengths through the inhomogeneous medium due to finite antenna apertures. Digital communication over such channels is usually limited to certain maximum transmission rates because the transmitted pulses appear smeared out at the receiver and thus require at least a certain minimum spacing to be distinguishable at the receiver. The performance criterion

which is, therefore, applied to the Signal Design problem is the minimization of intersymbol interference and the minimization of transmitter power required for a specified probability of received errors.

In order that numerical results may be obtained, a particular channel model is considered on which most of the discussion in the chapter is based. The method of approach is quite general, however, and the results obtained indicate the advantages to be gained by the proper design of signals.

1.5 Performance of Error Correcting Codes

Chapter VI deals with a quantitative analysis of the relative advantages of increases in transmitted power versus the use of error-correcting codes for binary symmetric channels. This analysis is subdivided into three major sections. The first section deals with the characterization of binary communications channels by the transitional or error probabilities, given the signal-to-noise ratio at the receiver and the modulation system used; the channel disturbances are restricted to additive white Gaussian noise.

The second section considers the determination of the bit error probability at the decoder output as a function of the channel error probability and the code characteristics. The analytically derived expression for Hamming codes is entirely new; the proof of the derivation is included as Appendix IV.

The final section presents, in graphical and tabular form, detailed results for the error rates and figures of merit for Hamming codes, based upon both constant transmitted binit rate and constant information binit rate. The results obtained by computer analysis for two of the shorter multiple-error correcting Bose-Chandhuri codes are also presented.

1.6 Recommendations

The final chapter of this report brings together the results and recommendations of the problems considered in this effort. Areas that look particularly promising are discussed in greater detail and specific recommendations for continued study and/or experimental phases are made.

CHAPTER II

CHANNEL CHARACTERIZATION

2.1 Introduction

The specification and design of a reliable communication system requires fairly accurate knowledge of the channel through which one desires to transmit signals. In the past a large variety of different types of channels have been used for radio wave propagation. A partial listing is given below:

a) Ground-wave systems

b) Line-of-sight systems

c) Systems employing reflection from the ionosphere

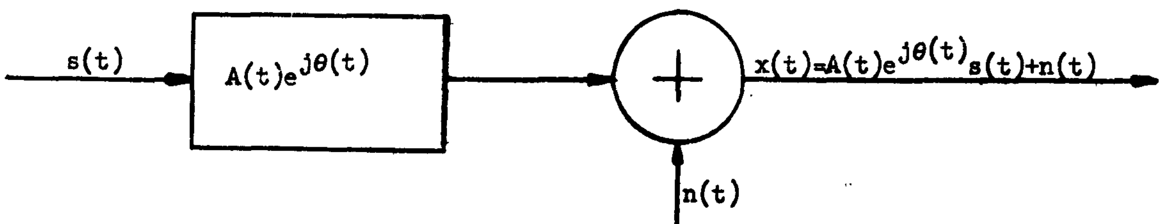
d) Ionospheric-scatter systems

e) Meteor-trail-reflection systems

f) Beyond-line-of-sight systems employing diffraction

g) Tropospheric-scatter systems

Although the transmission characteristics of these channels vary widely, the simple model shown in Fig. 2.1 can be used to analyze the performance of each of the channels. Note that the amplitude and phase distortion



CHANNEL MODEL

FIGURE 2.1

experienced by the transmitted signal, $s(t)$, is attributed to both the multiplicative disturbance, $A(t)e^{j\theta(t)}$, and the additive noise, $n(t)$.

This chapter presents a brief survey of the types of additive and multiplicative disturbances commonly encountered in typical channels.

2.2 Additive Disturbances

Additive noise is frequently assumed to be Gaussian. For many systems the Gaussian assumption appears to be a good one. Yet, there are many other systems (for example, those which employ ionospheric channels) in which the Gaussian assumption does not lead to a satisfactory prediction of system performance.

A literature survey on the statistical characterization of radio noise revealed that intensive work in this area has just begun, most of it having been carried out within the last four or five years. The initial measurements have been made at frequencies below 10 mc/s. Very little data is available above this frequency. The statistical data which has been obtained thus far pertains to the envelope of the noise as measured by a linear envelope detector, and not to the noise itself. Since a knowledge of the statistics of the envelope is not sufficient to deduce the statistics of the noise, much more statistical data remains to be taken before the noise can be adequately characterized so as to enable accurate prediction of system performance.

Radio noise falls into several categories. The most usual types of additive noise encountered are:

a) thermal noise

b) man-made noise

c) noise from precipitation, blowing snow or dust

d) noise from corona

e) atmospheric noise

Each of these types of noise is briefly discussed in the following sections.

2.2.1 Thermal Noise(1, 2, 3, 4)

From thermodynamical reasoning it can be shown that all materials which are capable of absorbing radiation are sources of thermal noise. In fact, good absorbers of radiation are good thermal noise sources while poor absorbers of radiation are poor sources of thermal noise. Hence, thermal noise is generated by the ground, the troposphere, the ionosphere, and extra-terrestrial sources.

While the ground may act as a good reflector of radio waves at glancing incidence, this is typically not true at steeper angles of incidence, particularly for vertical polarization. The two obvious ways of reducing ground noise (which is rarely serious below about 200 mc/s) are to limit the sensitivity of the antenna in the direction of the ground, and to increase the reflection coefficient of the ground. The former may be achieved by minimizing side lobes in the downward direction; the latter may be achieved by using an artificial ground plane of radial wires, or mesh, or in special cases by taking advantage of the very high reflection properties of sea water.

Under some circumstances, and particularly at wavelengths less then about 1.5 cm, the troposphere can act as an absorbing medium. The two atmospheric constituents responsible for this absorption are water vapor and oxygen.

VHF radio waves can, under certain circumstances, undergo significant absorption in the ionosphere; on these occasions the ionosphere will act as a source of thermal noise. Since the number of decibels of attenuation in the ionosphere at VHF is proportional to $\frac{1}{f^2}$, the ionosphere contribution to thermal noise tends to decrease rapidly with increasing frequency.

Extra-terrestrial thermal noise originates from the various galaxies, the sun, the moon, and the planets. Galactic noise imposes a very important limitation to communication systems in the HF and VHF bands (3 - 300 mc/s). The intensity of thermal noise generated by the sun varies considerably, especially in the VHF band, and during years of high sunspot number. The contributions due to lunar and planetary thermal noise are likely to be negligible compared to that of the sun.

2.2.2 Man-Made Noise[1, 5]

Man-made noise is generated by almost all types of electrical devices and machinery. Since it is almost always propagated along power lines or by groundwave, the propagation is not affected appreciably by ionospheric conditions. However, there is some experimental evidence that man-made noise may also be received from distant sources via ionospheric propagation.

The noise is usually impulsive in nature. When many sources are involved, the envelope probability density is similar to that of atmospheric noise. However, the dynamic range is usually considerably less than that encountered in atmospheric radio noise. The radiated energy often has strong components which extend far into the radio-frequency spectrum (up to tens of megacycles per second).

2.2.3 Noise From Precipitation, Blowing Snow or Dust, and Corona

The radio noise caused by precipitation, blowing snow, or blowing dust or sand is the result of charged particles actually hitting the antenna. These particles become charged as they move through the air, and as these contact the antenna, the charge is transferred to the antenna.

Corona noise is caused by the presence of a low, highly-charged cloud passing over the antenna, causing an actual corona discharge at the tip of the antenna. Not much is known quantitatively about the levels encountered under these two conditions. When these conditions have been observed at various noise recording stations, the level of the noise has increased on all frequencies up to 20 mc/s to the top of the recorder scale, which has been in several cases as much as 50 db above the level prior to the occurrence of the phenomenon.

2.2.4 Atmospheric Noise[1, 6, 7, 8]

The principal sources of atmospheric noise are the lightning discharges which occur during thunderstorms. Approximately 44,000 thunderstorms occur somewhere in the world every day. Due to these storms there occur on the average 100 lightning strokes per second. The amount of charge involved in a lightning stroke is about 10 coulombs and the peak current is in the region of 50,000 amperes. Lightning energy, like ordinary radio signals, reaches a receiver by all of the well-known mechanisms of propagation, including surface wave, tropospheric wave, and ionospheric sky wave. In addition, there is the whistler mode of propagation for frequencies below 35 kc/s in which the lightning energy is guided by the earth's magnetic lines of force up to distances half way around the world. The spectrum of the radiated energy covers a wide frequency range, from as low as a few cycles to tens of megacycles per second.

A typical amplitude probability density distribution of an atmospheric noise envelope is shown in Fig. 2.2[9]. The coordinates are plotted as noise level in decibels above the root mean square voltage versus the percentage of time that each level is exceeded. Rayleigh graph paper is

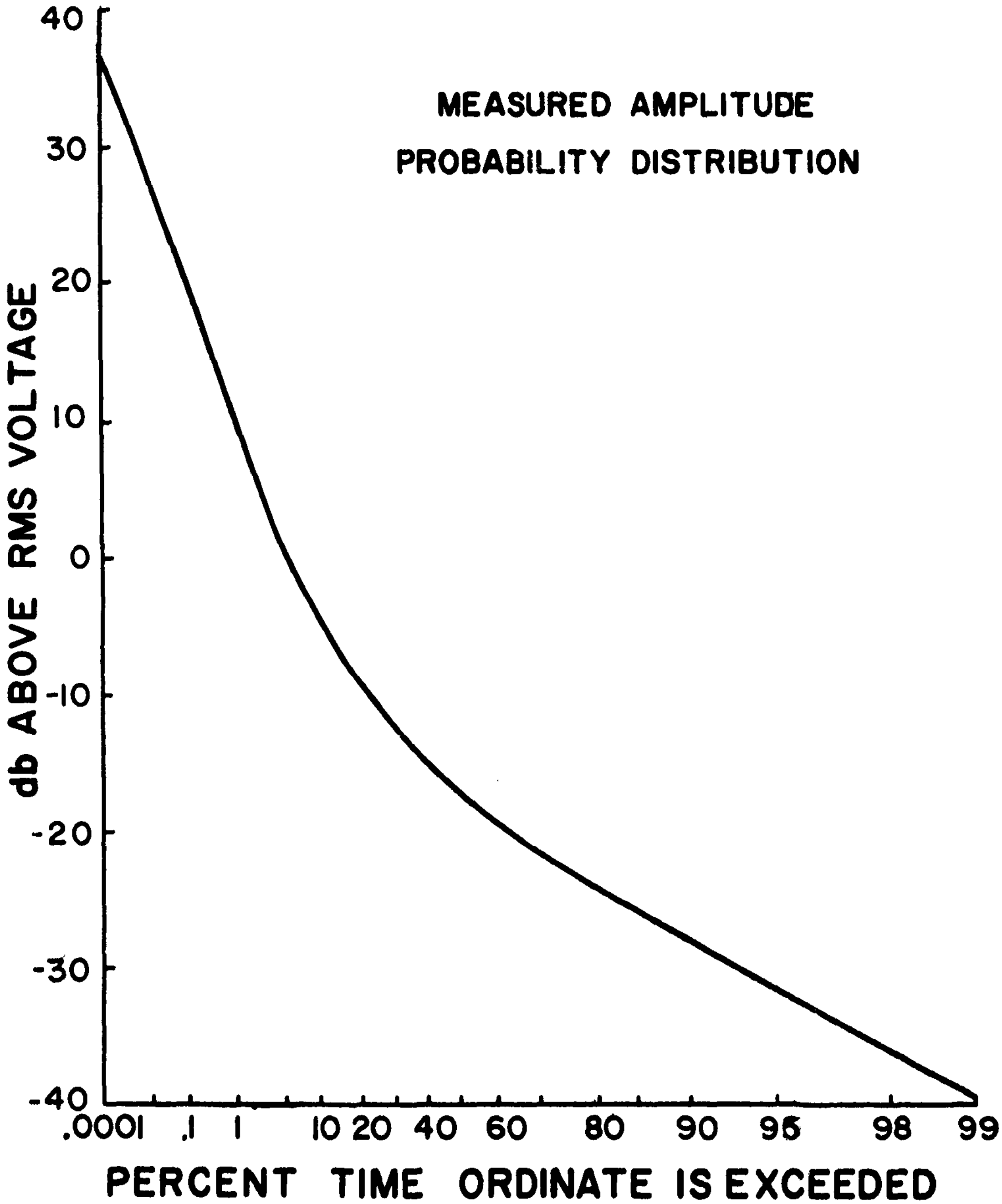


FIGURE 2.2
A TYPICAL AMPLITUDE PROBABILITY DENSITY DISTRIBUTION
OF AN ATMOSPHERIC NOISE ENVELOPE

used so that a distribution of the form

$$P(X \geq x) = e^{-x^m}$$

plots as a straight line with a negative slope of $\frac{1}{m}$. In particular, the Rayleigh distribution plots as a straight line with a slope of - 1/2.

The lower portion of the curve, representing low voltages and high probabilities, is composed of many random overlapping events, each containing only a small portion of the total energy. The Central Limit Theorem states that if several independent events of this type are superimposed, the sum tends rapidly to a Gaussian process as the number of components (of roughly equal power) is increased. Hence, we would expect the lower portion of the curve to approach a Rayleigh distribution since the envelope of a Gaussian process is Rayleigh.[10, 11] This is seen to be the case, the slope of the lower portion of the curve being very close to - 1/2.

The section representing very high voltages exceeded with low probabilities is, in general, composed of nonoverlapping large pulses occurring infrequently. From experimental measurements of atmospheric noise distributions, this section has been found to be well represented by a straight line on Rayleigh graph paper with values of m in the range from +0.1 to +0.4.[12]

On this graph paper, the remaining section of the distribution has been found to correspond quite closely to an arc of a circle tangent to the above two straight lines. The National Bureau of Standards has developed a graphical method for constructing the entire envelope amplitude probability distribution from only three measured statistical moments.[9]

The dynamic range of the distribution, as measured between the

0.0001 per cent and 99 per cent intercepts, has been observed to vary from a low of 59 db to a high of 102 db. An average dynamic range appears to be around 73 db. The variations in dynamic range for frequencies above 35 kc/s agree with expectations based on the distribution of distances to thunderstorms where it is apparent that small dynamic ranges will result if the range of distances to the effective thunderstorms is small. The above statement does not necessarily hold for frequencies below 35 kc/s because of the whistler mode of propagation.

The envelope amplitude distributions for the highest and lowest observed average power levels show a difference of 46 db between the root mean square values of voltage. The high-level curve was obtained on a day with a large number of local afternoon mountain thunderstorms while the low-level curve was obtained during the morning of a relatively quiet day.

It should be pointed out that the distributions mentioned above are strictly valid only for the bandwidth in which the measurements are made. Typical bandwidths used were on the order of 1100 cycles per second. The principal effects of reducing the predetection bandwidth are a reduction in the dynamic range with a greater and greater portion of the distribution curves becoming a straight line of slope equal to - 1/2. Measurements in an 0.2 cycle band yielded a Rayleigh distribution over the entire range measured. These results are reasonable since as the observing bandwidth is reduced, the energy from all the received impulses is spread out over a greater period of time with a resulting decrease in the amplitudes of the impulses.

Generally, the additive noise encountered on ionospheric channels is atmospheric noise. Montogmery has shown that in a binary narrow-band

frequency modulation system the errors can be calculated as one-half the probability of the noise envelope exceeding the carrier envelope. Hence, the envelope statistics described above can be used in calculating the probability of error for a narrow-band FSK system utilizing an ionospheric channel. Experimental curves have been obtained which overlap the theoretical curves quite closely. Fig. 2.3 shows the large discrepancies which can occur in system performance if Gaussian noise is assumed rather than atmospheric noise. For signal-to-noise ratios larger than 6 db the error rates experienced with atmospheric noise are much larger than those experienced with Gaussian noise.

2.2.5 Concluding Remarks

The Gaussian assumption is likely to be a good one for thermal noise internal to the receiving system, solar, lunar, planetary, and cosmic noises. In terms of frequencies, all noise above 150 mc/s can usually be assumed to be Gaussian. It should be pointed out that above 300 mc/s the thermal noise generated internally in the receiving system is usually the controlling noise. Between 30 and 150 mc/s the major noise is most often of galactic origin. Below 30 mc/s atmospheric noise and man-made noise predominate over the other types of noise for a greater percentage of the time. This is shown in Fig. 2.4.

2.3 Multiplicative Disturbances

Multiplicative disturbances are responsible for such phenomena as fading, dispersion, multipath, phase distortion, and time delay. Since these disturbances vary widely, depending upon the frequency of the transmitted radio wave, they are most easily discussed by making reference to the pertinent frequency bands.

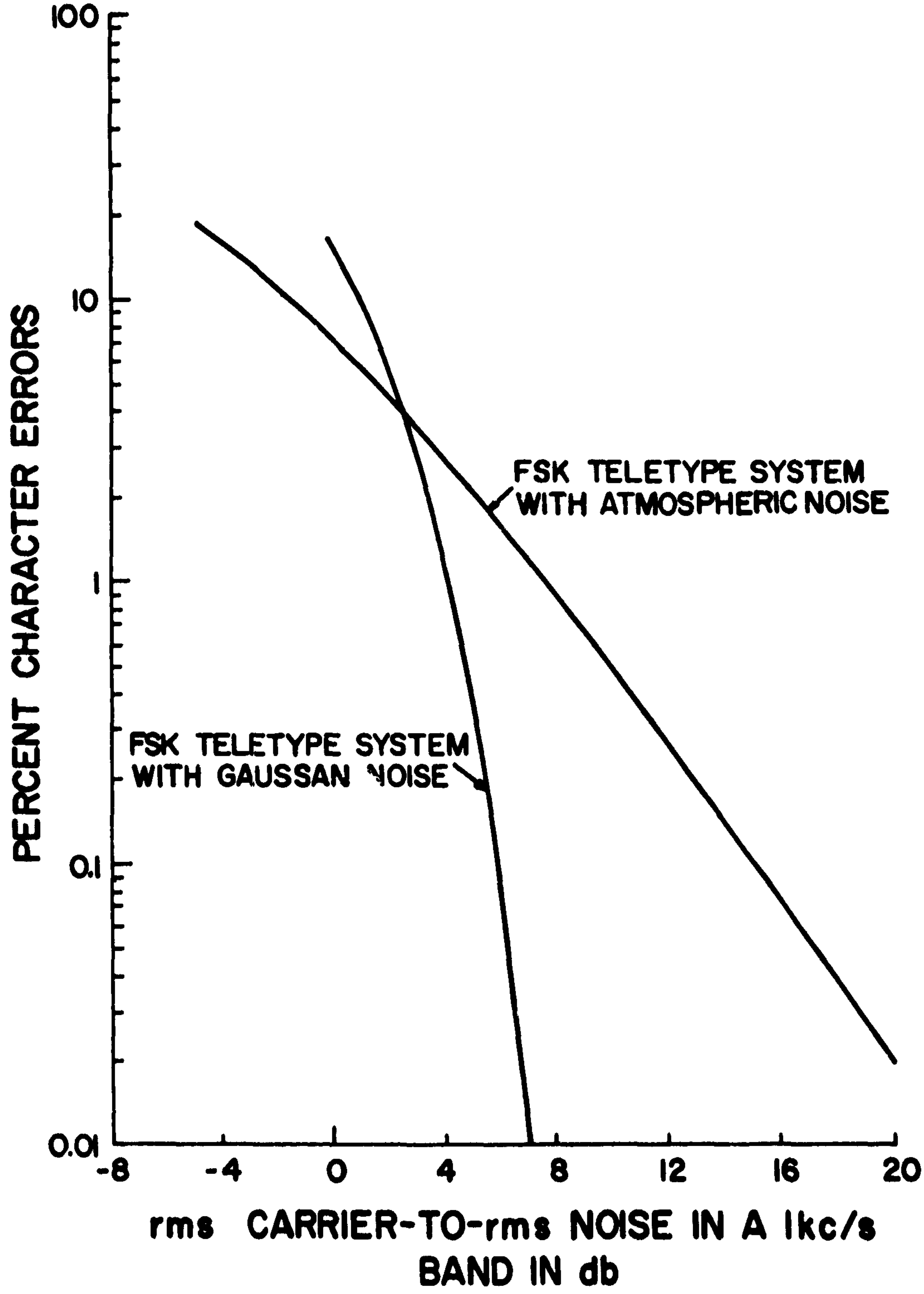


FIGURE 2.3
COMPARISON OF SYSTEM PERFORMANCE FOR GAUSSIAN AND
ATMOSPHERIC NOISE

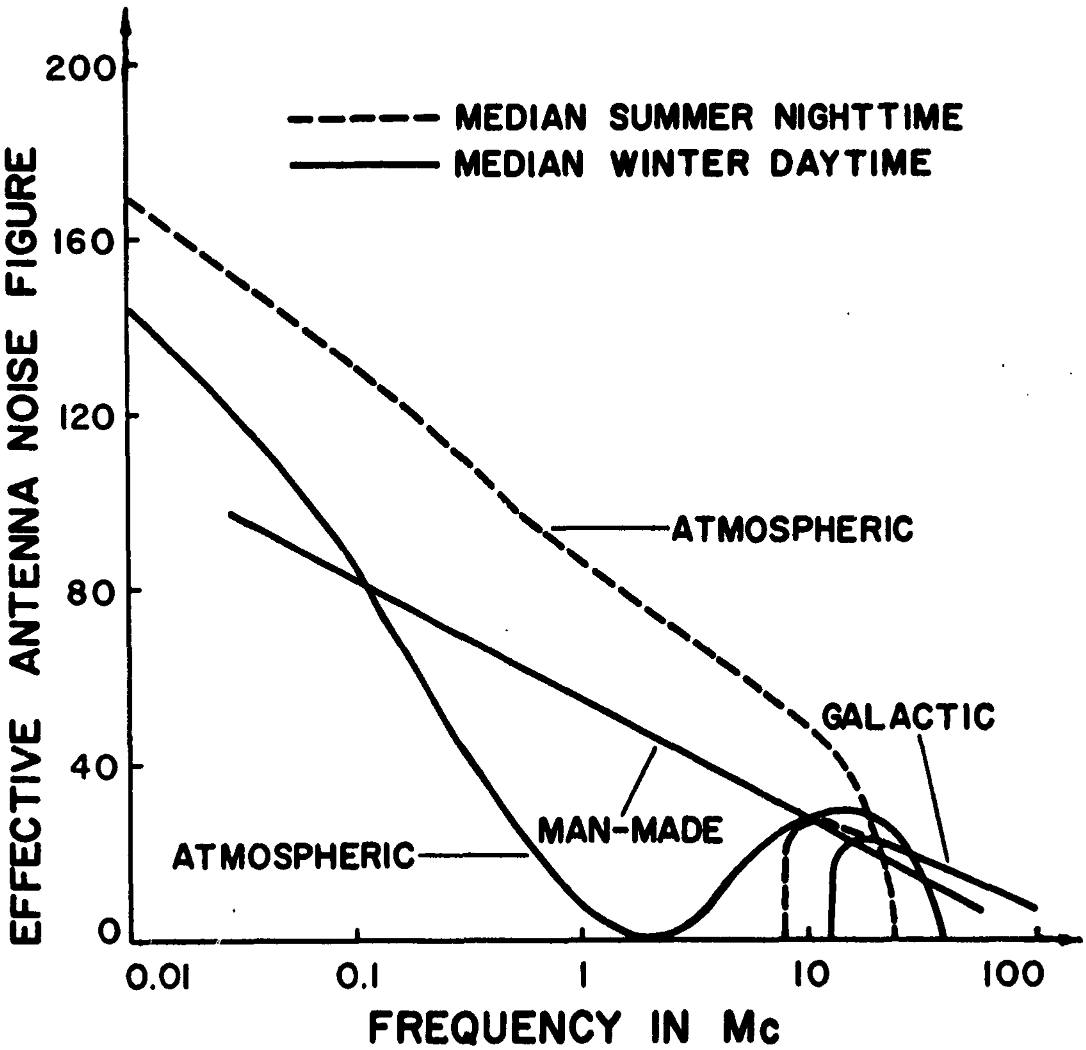


FIGURE 2.4

RADIO NOISE MEASUREMENTS DURING IGY

GUNBARREL HILL, COLORADO

2.3.1 3-30 kc/s VLF[1]

VLF propagation, occurring in the form of waveguide modes between the earth and the ionosphere, is often referred to as ducting. Propagation in the VLF range is characterized by low attenuation to very great distances, with great reliability and stability of transmission. Because of the large physical structure required for transmitting antennas (one wavelength is 30 kilometers at 10 kc/s) antennas are electrically small, and either costly or inefficient. The Q of many typical transmitting antenna systems in this frequency range limit the modulation bandwidth to less than 100 cps.

The amplitude of VLF signals is highly variable at short distances. The amplitude also has a tendency to change rapidly during the period of sunrise or sunset along the path. At these distances, the amplitude generally goes through a rapid maximum or minimum, before tending toward the more steady value characteristic of midday or midnight.

At distances beyond about 1000 km, attenuation is typically 2 to 4 decibels per 1000 km. Penetration of VLF energy into conducting earth or even sea water makes the frequency range useful for communication between buried antennas or submarines. The constancy of phase of the received signal at distances beyond about 500 km allows communication systems to use stored reference phase information.

VLF systems are commonly used for reliable long-range communication, navigational aids, and frequency and timing standards.

2.3.2 30-300 kc/s LF[1]

The LF spectrum is characterized by higher path attenuation, lower background noise levels, and more stable propagation time delays relative

to VLF paths. Transmitting power and antenna requirements are appreciably less than those of the usual VLF station, and in addition the bandwidths available are greater. The higher path attenuation results from the fact that as the frequency increases, the ionosphere behaves less and less as a sharp boundary. Hence, the radio waves reach the receiver only after they have penetrated into the ionosphere and lost energy in absorption.

The fading speed and the depth of fading depend on the frequency, the transmission distance, and the time of day. During the daytime the amplitude is substantially constant. The fading during the nighttime is much more irregular. The amplitude fluctuations are approximately Rayleigh but assume large values more often than would be expected on a Rayleigh distribution.

As with VLF the transmitting installations are characterized by their large physical size and high construction and maintenance costs. LF waves are not adversely affected during periods of ionospheric disturbance and the phase stability of transmission, permitting frequency comparison within a few parts in 10^{10}, makes possible long range radio navigation utilizing phase comparison between spaced phase-locked transmitters.

2.3.3 300 kc/s -- 3 mc/s MF[1]

The medium frequency range is a transition range in which the importance of the ground wave at the lower frequencies gives way to the importance of the sky wave at the higher frequencies. Ground-wave attenuation increases with frequency, so that in the higher part of the frequency range only short distance services are possible, especially over paths of poor conductivity.

Sky-wave propagation via the E and F regions of the ionosphere is important mainly only during the night hours; it is sometimes observable during daytime, but is usually highly absorbed in the D region of the ionosphere. Transmission in this frequency range, especially above about 500 kc/s, is very susceptible to absorption, and, even at night, sky waves are often attenuated below useful levels.

Because of the unreliability of the sky wave, the frequency range is probably most useful from the low end up to about 1 mc/s, where the ground wave enables broadcast coverage out to several hundred miles.

2.3.4 3-30 mc/s HF[1]

HF propagation is characterized by the ability of high frequency waves to penetrate the lower ionosphere and be reflected from the F region of the upper ionosphere. Absorption is of minor concern and transmission loss, even for a long transmission distance (10,000 km or more), may be quite low. Useful signal-to-noise ratios are obtainable out to very great distances with very low power and simple antennas.

Because of considerable variability of propagation conditions, transmission is very unreliable. The consequence is that for optimum results the transmitter must be capable of changing to four or five different frequencies, hoping that one will work.

Multipath is a serious problem. At HF there are a large number of possible propagation paths with multipath time delays ranging from a few microseconds to a few milliseconds. Multipath propagation imposes a limit on keying speeds in digital systems since if the multipath delays are such that during the sampling time there is still energy arriving from the preceding pulse, there is a high probability of error. Pulse durations should be somewhat more than twice the length of the greatest

significant multipath delay. At HF it usually occurs in the range of from 1 to 5 milliseconds for paths longer than about 100 km. Multipath can be reduced by operating at as high a frequency as possible. At the MUF (maximum usable frequency) only one geometric mode is possible.

In addition to multipath effects, dispersion may cause important distortion of the transmitted waveform in the case of short pulses. The first-order effect is a lengthening of the pulse. Under worst conditions pulses on the order of 1 microsecond in width are stretched to 13 microseconds.

Both fast and slow fading are observed in connection with the transmission of HF radio waves. The fast fading is usually due to the interference of two or more unresolved propagation modes. The slow fading is attributable to variations in absorption, or changes in the effective gains of the transmitting and receiving antennas resulting from changes in the angles of departure and arrival of the signals. In fast fading, fades tend not to occur simultaneously at nearby frequencies. This effect is called selective fading. Slow fading tends to occur across a broad band of frequencies and is referred to as flat fading. The fading distributions of the amplitudes approximate the well-known Rayleigh distribution when the wave arrives via several modes with approximately equal amplitude and randomly varying phases. Fading rates from 1 cps to 15 cps are commonly observed.

Phase and frequency stability is very poor at HF. This imposes genuine limitations on minimum modulation excursions for FSK and PSK systems. Phase perturbations up to 140° and frequency shifts up to 50 cps have been observed.

In spite of the difficulties mentioned above, there is a great density of radio services in the high frequency range. A substantial part of the

world's frequency assignments are concentrated in this small fraction of the whole spectrum.

2.3.5 30-60 mc/s VHF Ionospheric Scatter[1]

Irregularities in electron density in the lower ionosphere give rise to incoherent scattering of radio waves in the frequency range between 30 and 60 mc/s. Reliable transmission is obtained in the 1000 to 2000 km distance range. The scattered radio waves are extremely weak and system losses ranging between 140 and 210 db are commonly experienced. Typically, ionospheric scatter suffers around 150 db more loss than does ionospheric reflection. To compensate for the large losses, extremely large high gain antennas are employed.

Fading is observed at rates varying from 0.2 to 3 cps. During most of the day the envelope fading is approximately Rayleigh distributed, though amplitude distributions indicate peaks from meteor reflections during the night hours. The fading characteristics depend upon the beamwidth of the antennas employed. For a 60° horizontal beamwidth, the fading rate has been observed to be four to five times greater than for a 6° beamwidth system, and the depth of fading several decibels greater for the wide beam system.

Multipath caused by reflections from meteor trails usually displays delays varying from 6 microseconds to 1 millisecond. The time delays of multipath due to auroral ionization are typically between 0.1 and 4 milliseconds. During times of high solar activity, distant ground backscatter can be propagated by the F_2 layer of the ionosphere resulting in delays up to 80 milliseconds. Typically, the delays from this source are between 12 and 60 milliseconds. Because of the intersymbol interference caused by such multipath, an upper bound is placed on the keying rate of digital systems.

As with HF signals, the frequency and phase stability is poor. At 50 mc/s the expected Doppler shift is 6 kc/s. The large Doppler shifts are due mainly to meteor reflections. Often these signals are stronger than the direct scatter signal. Large phase shifts are experienced During night hours 180° shifts occur approximately 1 per cent of the time. Instantaneous phase shifts of 90° occur about 0.2 per cent of the time.

2.3.6 30-300 mc/s Meteor Scatter at VHF(1)

Each day billions of meteors enter the earth's atmosphere. In burning up they form long columns of ionized particles. These columns diffuse rapidly and usually disappear within a few seconds. However, during their brief existence the ionized columns will reflect radio signals, giving rise to what is called meteor scatter or meteor propagation.

Meteor-burst communication systems are basically weak-signal systems because the signal loss associated with the meteor-trail reflection is relatively high. For example, a typical system operating at 50 mc/s over a 1300 km path with a transmitter power of 2 kw was commonly set to transmit messages whenever the signal at the receiver exceeded 2×10^{-14} w (2 microvolt open-circuit voltage for a 50 ohm source). This corresponds to a system loss of 170 db. Of this total about 90 db represents the attenuation associated with the length of the transmission path and 80 db the scattering loss. Under similar circumstances ionosphere scatter propagation would exhibit a system loss of the order of 180 db. Messages are transmitted only during the brief intervals when meteor propagation is present.

At 50 mc/s Doppler shifts as large as 5 kc have been observed.

2.3.7 50-10,000 mc/s Tropospheric Scatter(1)

Tropospheric scatter results from irregularities in the refractive index of the atmosphere. The signals are much weaker than the VLF and LF

signals which employ tropospheric duct propagation. They are very reliable and are found to be present on a given path with substantially the same average intensity day and night, week in and week out, regardless of surface meteorological conditions. They also exhibit rapid fading, characteristic of multipath transmission.

The dominant feature of tropospheric scatter signals is their rapid fading. If a constant intensity signal is emitted at the transmitter, the level of the received signal varies erratically in time with an amplitude distribution that often closely approximates the Rayleigh law. Occasions have occurred, however, when this is not the case. Spectra of the rapid signal fluctuations closely approximate a Gaussian distribution.

Measurements made at frequencies of 400, 3,670, and 5,050 mc/s utilizing antennas with several degrees beamwidth indicate that time delays of about 1 microsecond at distances of about 200 miles can be expected. At 3,700 mc/s 1 microsecond pulses were not substantially widened after transmission over distances up to 285 miles. It appears that modulation bandwidths of several megacycles may be used.

2.3.8 Space Communications

The frequency of the transmitted signal must be above 30 mc/s to enable the radio waves to penetrate the ionosphere. Between 30 to 60 mc/s the wave experience considerable amplitude and angular scintillations. Above 100 mc/s radio waves propagate into space fairly well. Severe fading has been noticed at certain frequencies and multipath has been observed which cannot be explained by current theories. Many measurements are currently being made to understand the radio wave propagation involved in space communication.

2.3.9 Concluding Remarks

The results of this section are summarized in Table 2-1. The tabulated disturbances and propagation characteristics must be taken into account in developing a communication system. The remainder of this report is an effort in that direction. In particular, the capacity of a Rayleigh fading channel, the design of systems when the statistics of the additive noise are unavailable, the process of signal design and selection, and the use of error-correcting codes are discussed.

TABLE 2-1

CHANNEL CHARACTERIZATION

FREQUENCY BAND	METHOD OF PROPAGATION	TYPICAL DISTANCES	MULTIPATH	PHASE STABILITY	RELIABILITY	TYPICAL MODULATION BANDWIDTHS
3-30kc/s	ducting	5-20Mm*	no	good	very good	20-150 cps
30-300kc/s	ducting	1-5Mm	no	good	good	250 cps
300 kc/s- 3 mc/s	transition region between ducting and ionospheric reflection	200 miles	no for ground wave, yes for sky wave	good for ground wave, poor for sky wave	good for ground wave, poor for sky wave	2-75 kc/s
3-30mc/s	ionospheric reflection	1-10Mm	yes	very poor	poor	3 kc/s
30-60mc/s	ionospheric scatter	1000-2000 km**	yes	very poor	fair	5 kc/s
50-10,000mc/s	tropospheric scatter	100-1000 km	yes	very poor	good	10 mc/s

* Mm = megameter

** km = kilometer

CHAPTER III

CAPACITY OF THE RAYLEIGH FADING CHANNEL

3.1 Introduction

As was discussed in Chapter II many of the communication channels commonly used experience Rayleigh type fading. In this chapter the following assumptions are made concerning the parameters of the channel model given in Fig. 2.1.

1) The multiplicative disturbance $A(t)e^{j\theta t}$ is equal to A, where A is a random variable, Rayleigh distributed with parameter $\frac{\sigma}{\sqrt{2}}$

$$p(A) = 2Ae^{-A^2/\sigma^2} \qquad A \geq 0$$

$$= 0 \qquad A < 0 \qquad (3\text{-}1)$$

2) The additive noise n(t) is assumed to be a stationary Gaussian random process with zero mean and uniform power spectrum over information bandwidth W. If $E\,[n^2(t)] = N$, then the noise spectrum is $G_n(f) = \frac{N}{2W} = N_o$

3) The signal s(t) is a sample function from a stationary random process and has a finite power P. The power spectrum of the signal is $G_s(f)$ and the signal is bandlimited to W cycles per second.

In this chapter the channel capacity of the Rayleigh fading channel is derived. The results are then used to evaluate β, the required received energy per information bit received in the presence of a given Gaussian noise spectral density.

3.2 Calculation of Channel Capacity

Capacity is defined as the maximum information, on the average, that an observer at the output of the channel can obtain about a signal transmitted from the channel input. The maximization of the information

rate being carried out through the variation of the input signal characteristics, i.e., the encoding process. Capacity C, may therefore be expressed as:

$$C = \max_{\{s(t)\}} I(S/X) \tag{3-2}$$

To solve for the capacity of the Rayleigh fading channel using the above Eq. (3-2) is a very difficult non-linear problem.

Another expression for the capacity is given by Fano[14]

$$C = \max_{\{G_s(f) \geq 0\}} I(S/X) \tag{3-3}$$

It is easier to solve for the capacity of the Rayleigh fading channel using the above Eq. (3-3) since maximizing over the power spectrum of the signal is maximizing under less restrictive conditions.

Using Eq. (3-3) for the conditional information rate (assuming A the attenuation factor is known) results in

$$\max_{\{G_s(f) \geq 0\}} I(S/X,A) = \max_{\{G_s(f) \geq 0\}} \int_{f_o}^{f_o+W} \log_2 \left\{ 1 + \frac{A^2 G_s(f)}{G_n(f)} \right\} df \tag{3-4}$$

Since the attenuation factor A is a random variable it is necessary to average over all possible values of the random variable. Thus,

$$C = \max_{\{G_s(f) \geq 0\}} I(S/X) = \max_{\{G_s(f) \geq 0\}} \int_{-\infty}^{\infty} I(S/X,A)\, p(A) dA \tag{3-5}$$

$$= \max_{\{G_s(f) \geq 0\}} \int_{f_o}^{f_o+W} df \int_0^{\infty} \log \left\{ 1 + \frac{A^2 G_s(f)}{G_n(f)} \right\} \frac{2A}{\sigma^2} e^{-A^2/\sigma^2} dA \tag{3-6}$$

Integrating with respect to A, (see Appendix I for details)

$$C = \max_{\left\{ G_s(f) \geq 0 \right\}} (\ln 2)^{-1} \int_{f_o}^{f_o+W} \mathrm{Ei}\left\{ \frac{-G_n(f)}{\sigma^2 G(f)} \right\} \exp\left\{ \frac{G_n(f)}{\sigma^2 G_s(f)} \right\} df \tag{3-7}$$

where

$$\mathrm{Ei}(t) = \int_{-\infty}^{t} \frac{e}{u}^{u} du = e^t \sum_{k=1}^{\infty} \frac{(k-1)!}{t^k} \tag{3-8}$$

$$= -\ln \frac{-1}{\gamma t} + \sum_{k=1}^{\infty} \frac{t^k}{k\, k!} \qquad t < 0 \tag{3-8'}$$

$$-\mathrm{Ei}(-t) = e^{-t} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k-1)!}{t^k} \qquad t > 0 \tag{3-9}$$

Applying calculus of variations to maximize the above integral with respect to $G_s(f)$ yields

$$\frac{\partial}{\partial G_s(f)} \left[\exp\left\{ \frac{G_n(f)}{\sigma^2 G_s(f)} \right\} \mathrm{Ei}\left\{ \frac{-G_n(f)}{\sigma^2 G_s(f)} \right\} + \lambda\, G_s(f) \right] = 0 \tag{3-10}$$

where λ is the Lagrange multiplier for the power constraint,

$$2 \int_{f_o}^{f_o+W} G_s(f) df = P$$

Carrying out the above differentiation and simplifying results in

$$\lambda \sigma^2 G_s^{\,2}(f) - \sigma^2 G_s(f) - G_n(f) \exp\left\{ \frac{G_n(f)}{\sigma^2 G_s(f)} \right\} \mathrm{Ei}\left\{ \frac{-G_n(f)}{\sigma^2 G_s(f)} \right\} = 0 \tag{3-11}$$

From this result it is observed that if the additive noise power spectrum is uniform over a bandwidth W, then the power spectrum of the signal s(t) must also be independent of f. The signal power spectrum therefore equals,

$$G_s(f) = \frac{P}{2W}$$

Therefore, it has been proved that in order to transmit at a maximum rate through a Rayleigh fading channel the input signal s(t) must be from a stationary process with uniform spectral density. It is shown in Appendix II that the input signal must also be Gaussian with zero mean.

The capacity of the Rayleigh fading channel is therefore

$$C = -(\ln 2)^{-1} W \exp\left\{\frac{N}{\sigma^2_P}\right\} Ei\left\{\frac{-N}{\sigma^2_P}\right\} \tag{3-12}$$

3.3 Determination of the β Factor

One way to compare communication systems is to compare their efficiency in terms of β, the received signal energy required per information bit received in the presence of a given uniform Gaussian noise spectral density[15]

$$\beta = \frac{E_{min}}{2N_o} \tag{3-13}$$

E min = minimum received energy required per information bit received.

N_o = noise spectral power density.

Equivalently, β may be expressed as

$$\beta = \frac{P\ min}{2N_o H} \tag{3-14}$$

P = minimum received power required per bit of information received.

H = rate of received information (bits per second).

Letting H be equal to the maximum received information rate, on the average, for the Rayleigh fading channel one obtains for β

$$\beta = \frac{\sigma^2 P_{min}}{\left(\frac{N}{W}\right) \left[-(\ln 2)^{-1} W \exp\left\{\frac{N}{\sigma^2 P_{min}}\right\} \mathrm{Ei}\left\{\frac{-N}{\sigma^2 P_{min}}\right\} \right]} \tag{3-15}$$

$$\beta = \frac{-\ln 2 \;\; \sigma^2 P_{min} \exp\left\{ \frac{-N}{\sigma^2 P_{min}} \right\}}{N \; \mathrm{Ei}\left\{\frac{-N}{\sigma^2 P_{min}}\right\}} \tag{3-15'}$$

The lower bound on β occurs as the received signal-to-noise power ratio goes to zero. (See Appendix III and graph 3.2 for proof.) This lower bound on β is shown to be given by

$$\beta_{min} = \ln 2 \tag{3-16}$$

Note that this lower bound on β is the same as that obtained by Sanders for the single path channel having no fading. In fact the lower bound on β will always equal ln2 and is independent of the type of probability density function for the attenuation factor A. To see why this is so one notes that the conditional lower bound on β (conditional in the sense that A is fixed) is independent of the value of A. Averaging over the different values of A will therefore yield the same value as for the unity gain channel.

Another way of defining β in order to bring out the dependence of the channel is to define β as the minimum required energy transmitted per bit of information received. Under this definition the lower bound on β can be shown to be

$$\beta_{min} = \frac{\ln 2}{\sigma^2} \tag{3-17}$$

where this lower bound on β is obtained by letting the signal-to-noise power ratio approach zero. Since $\sigma^2 \leq 1$ for all passive channels, the lower bound of β is increased by a factor σ^{-2} over that of the single path with gain equal to unity. In other words, assuming the transmission rate is the same, the minimum power that must be transmitted is increased by σ^{-2} in order to maintain the same probability of error.

3.4 Discussion of Results

The capacity of the Rayleigh fading channel is a function of the information bandwidth W and of the ratio of received signal power to the received noise power,

$$C = \frac{-W}{\ln 2} \exp\left\{\frac{N}{\sigma^2 P}\right\} Ei\left\{\frac{-N}{\sigma^2 P}\right\}$$

It should be noted that if

$$\frac{\sigma^2 P}{N} \gg 1$$

then
$$Ei\left\{\frac{-N}{\sigma^2 P}\right\} \approx -\ln\left[\frac{\sigma^2 P}{\gamma N}\right] \tag{3-18}$$

where $\gamma = 1.781072$

The capacity may therefore be approximated by

$$C \approx W \log \frac{P'}{N} \tag{3-19}$$

where $P' \equiv \sigma^2 P$

Comparing the above equation with that for the unity gain channel one obtains

$$C = W \log\left(1 + \frac{P'}{N}\right) \approx W \log \frac{P'}{N} \tag{3-20}$$

If the signal-to-noise ratio $\frac{\sigma^2 P}{N} << 1$ then

$$C \approx \frac{W}{\ln 2} \left\{ \frac{\sigma^2 P}{N} \right\} = \frac{W}{\ln 2} \quad \frac{P'}{N} \tag{3-21}$$

However, the unity gain channel having a signal-to-noise ratio

$\frac{P'}{N} << 1$, has the capacity

$$C = \frac{W}{\ln 2} \ \ln \left\{ 1 + \frac{P'}{N} \right\} \approx \frac{W}{\ln 2} \quad \frac{P'}{N} \tag{3-22}$$

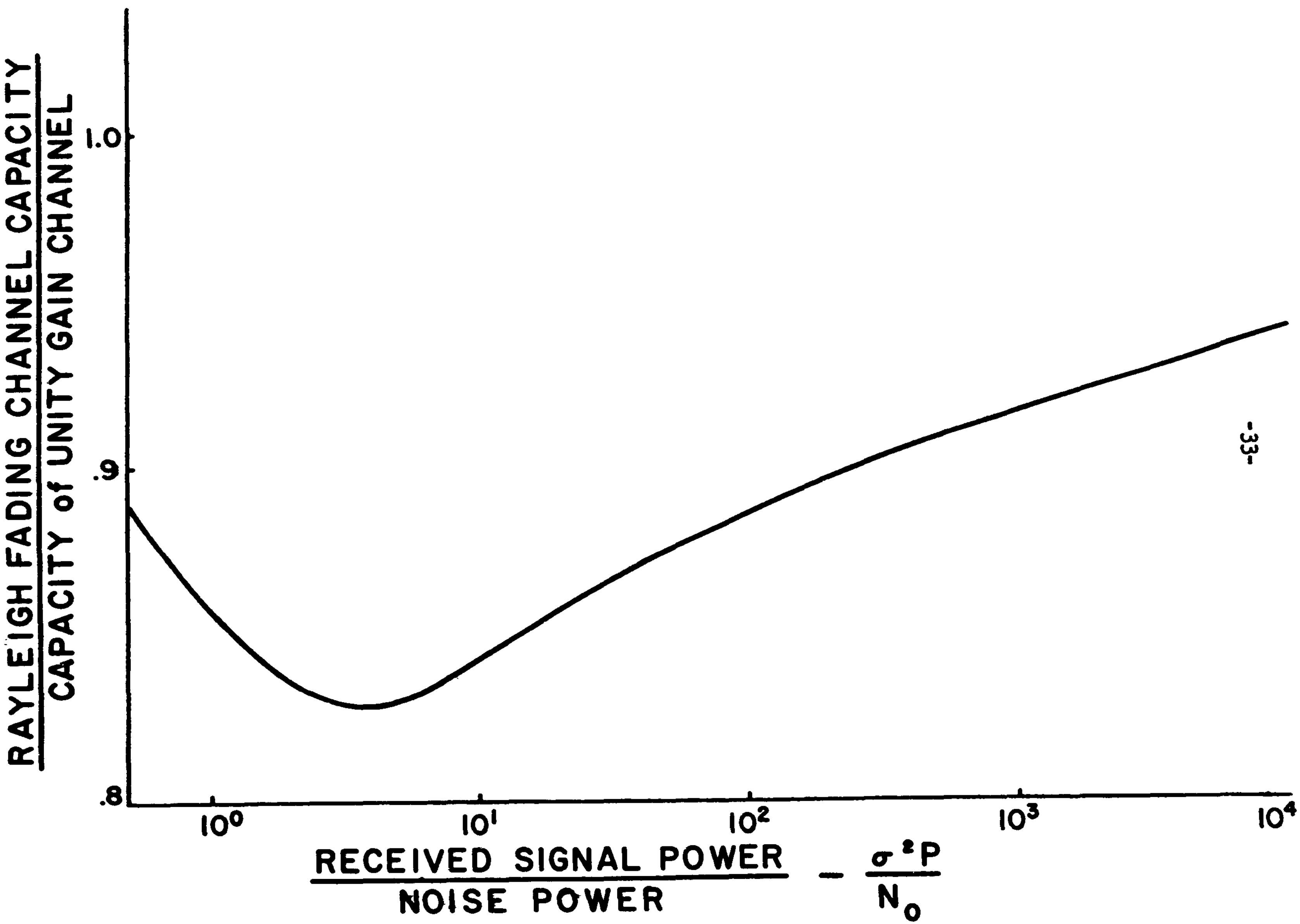
Hence, for small signal-to-noise ratios, i.e., $\frac{P'}{N} << 1$, the capacity of the Rayleigh fading channel is identical to that of the unity gain channel.

From Fig. 3.1 it is observed that the capacity of the Rayleigh fading channel is never less than 83% of the capacity of the unity gain channel.

It should be noted that the channel variance σ^2 can be determined experimentally by transmitting a known carrier $\sin \omega_o t$, and measuring the average power at the receiver.

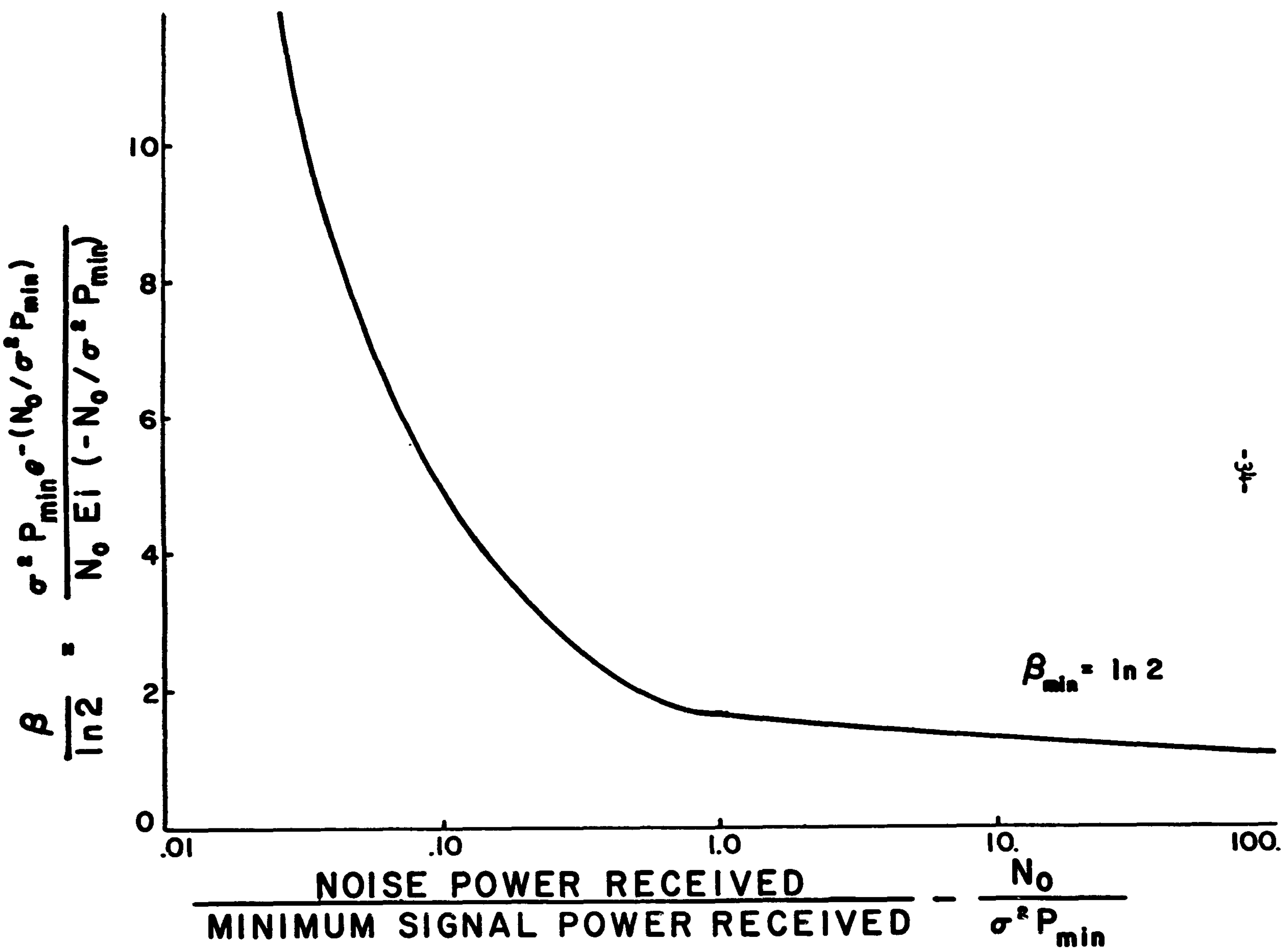
Fig. 3.2 illustrates that $\beta = \frac{W\,E_{min}}{N}$ is a monotonically decreasing function of the received noise to the received signal power ratio. Qualitatively this implies that the received signal energy required per information bit transmitted (assuming that the bandwidth is constant) varies as

$$E_{min} \propto N_o^k \qquad\qquad k < 1$$



RAYLEIGH FADING CHANNEL CAPACITY DIVIDED BY THE CAPACITY OF UNITY GAIN CHANNEL AS A FUNCTION OF SIGNAL-TO-NOISE POWER RATIO AT THE RECEIVER

FIGURE 3.1



EFFICIENCY FACTOR β, VS. THE RATIO OF NOISE POWER RECEIVED TO THE MINIMUM POWER RECEIVED

FIGURE 3.2

CHAPTER IV
NONLIKELIHOOD DETECTION THEORY
PART I GENERAL THEORY

4.1 Introduction

The problem of detection of a signal in noise of known statistical properties has been investigated thoroughly in the past. However, these methods are completely inapplicable and inappropriate whenever these noise statistics are unknown.

In this investigation, a detection criterion based on the methods of non-parametric statistics is utilized which permits the design of detectors on the basis of much less a-priori information. Several detectors based on this detection criterion are investigated and their properties obtained. A comparison is made between these new (non-likelihood) detectors and the optimum (likelihood) detectors on the basis of information efficiency. Also, a practical design procedure is formulated for the design of these new (non-likelihood) detectors.

4.2 Statement of the Problem

Given a signal immersed in noise of unknown distribution function, a detector is to be designed based on a detection criterion that does not require knowledge of the noise and of the mixture of signal and noise probability densities.

4.3 Inadequacy of Present Methods

Detectors which determine the presence or absence of a signal in noise have been investigated extensively in the past. These investigations, however, have been based on the assumption that a great amount of a-priori

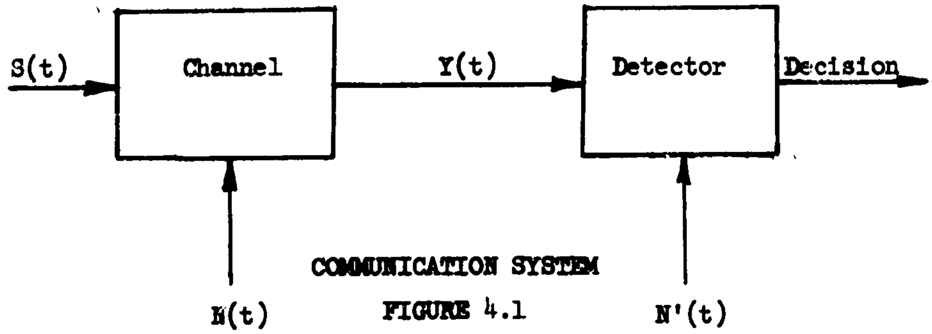
information is available concerning the probability densities of the noise and of the mixture of signal and noise. These detectors are based on the likelihood ratio.

However, these likelihood (optimum) detectors are completely inadequate and inappropriate whenever these noise probability densities are not known. This is so, since these detectors are optimum <u>only</u> for a particular pair of noise and mixture of signal and noise probability distributions for which they have been designed. In general, the probability of error (reliability of transmission of information) of the likelihood detectors depends on the functional form of these distributions. Therefore, if a likelihood detector which is optimum for a particular pair of probability distributions is used in another situation in which the distributions are different from the pair of distributions for which the detector is optimum, then it is possible and quite probable that the probability of error of the detector (unreliability of transmission) may increase to such an extent as to make the detector completely inapplicable. Moreover, due to this lack of <u>a-priori</u> information of the probability distributions, it is not possible to predict and evaluate theoretically the performance of these likelihood detectors. Hence, the likelihood detectors are inappropriate whenever there is incomplete information concerning the functional form of the underlying distributions.

4.4 The Non-likelihood (Non-parametric) Detection Criterion

In this investigation a detection criterion is used which leads to the design of detectors on the basis of much less <u>a-priori</u> information. These detectors, hereon called non-likelihood detectors, are based on statistical tests known in the statistical literature as non-parametric statistical tests.

In order to state this detection criterion we will introduce some assumptions and notation:



COMMUNICATION SYSTEM

FIGURE 4.1

Where S(t) is the signal, N(t) and N'(t) are sample functions of the noise random process $\{N(t)\}$

It is assumed that:

1) $\{Y(t)\}$ is a stationary continuous parameter stochastic process Since $\{Y(t)\}$ is identical to $\{N(t)\}$ when the signal is absent, it can be concluded that $\{N(t)\}$ is also stationary;

2) It is possible to obtain n independent samples Y_1, Y_2, ... Y_n from the sample function Y(t) of $\{Y(t)\}$;

3) There is available a sample function N'(t) from the stationary continuous stochastic process $\{N(t)\}$ of the noise;

4) It is possible to obtain m independent samples Y_{n+1}, Y_{n+2}, ... Y_{n+m} from the sample function N'(t).

On the basis of the samples Y_1, ..., Y_n and Y_{n+1}, ..., Y_{n+m} a decision procedure for detecting signals in noise is formulated by testing

H'_o: cumulative distribution function (cdf.) of Y_i is $P_o(Y)$ i=1, ..., n+m signal is absent, against

H'_1: cdf. of Y_i is $P_z(Y)$ i=1, ..., n and the cdf. of Y_{n+j} is $P_o(Y)$ for j=1, ..., m, signal is present.

where $P_o(Y)$ is the distribution function of any of the data elements (since $\left\{Y(t)\right\}$ is stationary) when signal is absent, and $P_z(Y)$ is the cumulative distribution function for any data element when the signal is present. Note that $P_z(Y)$ depends both on Y and on the signal-to-noise ratio z.

The above decision procedure simply states that: if the signal is absent then the cdf. of the Y_i, is $P_o(Y)$ and must be the same as the cdf. of the Y_{n+j}'s since both sets of observations were obtained from sample functions of the same continuous stochastic process $\left\{N(t)\right\}$. If the signal is present, then the cdf. of the Y_i's is $P_z(Y)$ which is not the same as the cdf. $P_o(Y)$ of the Y_{n+j}'s.

In a practical case, the sample function $N'(t)$ of the noise process $\left\{N(t)\right\}$ must be obtained from the noise entering the receiver during a time that no information is transmitted (signal absent). If the noise process is stationary then $N'(t)$ can be obtained once and for all before the transmission of information begins. From $N'(t)$, the m samples will then be obtained and stored in the receiver, to be compared later with the n samples obtained from $Y(t)$. If though the noise random process is not stationary, then, before the transmission of information commences, one obtains the m samples from the noise entering the receiver and uses them only for as long as the noise random process remains fairly stationary. Whenever the noise process varies considerably then the transmission of information must be interrupted for a sufficient time to enable one to obtain a new set of m samples to be used subsequently. If the noise process variations are of a permanent nature, a periodic sampling of the noise is necessitated. During sampling, the transmission of information must cease to permit the

acquisition of the m samples from the noise entering the receiver.

A practical example of a stationary type of noise is the case of continuous jamming with stationary noise. In this case, the m samples need be obtained only once, prior to commencing the transmission of information. A practical case of non-stationary noise is the case of on-off jamming where the jamming is on or off for periods comparable to the sampling interval required to obtain the n samples. In this case two sets of m samples must be available, one to be obtained and used when the jamming noise is off and the other set to be obtained and used when the jamming noise is on.

The theory of non-likelihood detection would be useful if it satisfies the following requirements: 1) it suggests the structure of the detection system; 2) it specifies procedures for evaluating the performance of such systems (information rate, probability of error); and 3) it specifies techniques of system comparison. It will be seen subsequently that the non-likelihood theory of detection does satisfy all of the above requirements.

4.5 General Properties of Non-parametric Detectors

In this investigation a restriction of the level of generality will be made by considering the detection of weak signals in noise. This means that the peak-signal-to-rms noise ratio and thus z is assumed to be very close to zero. This is appropriate since the weak signal case is the most troublesome and least amenable to solution and the case one usually desires to solve in practice. This is also expedient since it simplifies the analytical expressions found.

Many of the non-parametric detection test statistics satisfy the following properties:

1) The non-parametric detection statistic U_{mn} (the subscript mn is to show dependence on the samples m and n) is asymptotically Gaussian under H'_o (no signal). The mean and standard deviation of this limiting distribution are denoted by $E_o[U_{mn}]$ and $\sigma_o[U_{mn}]$, respectively,

2) U_{mn} is asymptotically normal under H'_1 (signal present). The mean and standard deviation of this limiting distribution are denoted by $E_z[U_{mn}]$ and $\sigma_z[U_{mn}]$, respectively;

3)
$$\lim_{z \to o} \frac{\sigma_z^2 [U_{mn}]}{\sigma_o^2 [U_{mn}]} = 1 \tag{4-1}$$

4)
$$E_z[U_{mn}] = E_o[U_{mn}] + z \left. \frac{dE_z[U_{mn}]}{dz} \right|_{z=o} + O(z^2) \tag{4-2}$$

5)
$$\lim_{z \to o} \left[\frac{dE_z[U_{mn}]}{dz} \Big/ \sigma_o[U_{mn}] \right]^2_{z=o} = \frac{Kmn}{m+n} \tag{4-3}$$

where K is a constant independent of m, n, z and defined by Eq. (4-3); K depends only on $P_o(Y)$ and $P_z(Y)$.

6)
$$\left. \frac{dE_z[U_{mn}]}{dz} \right|_{z=o} \neq o \tag{4-4}$$

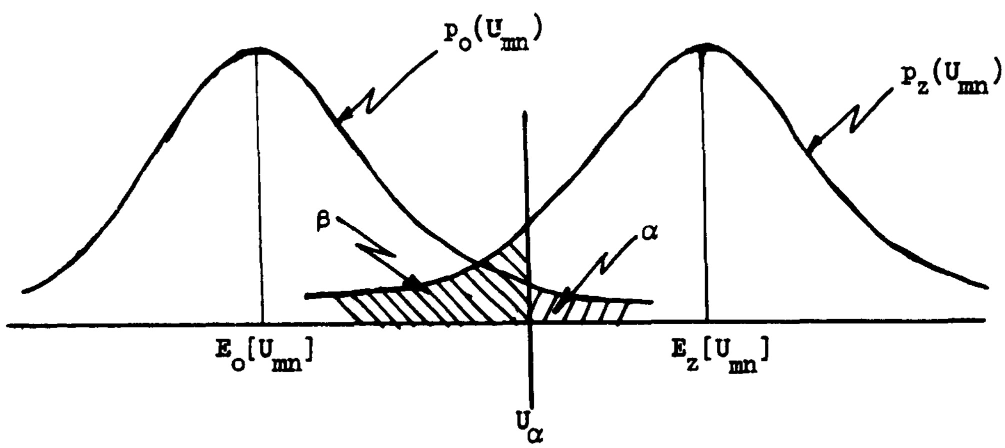
7)
$$\lim_{\substack{m \to \infty \\ n \to \infty}} \sigma_o^2[U_{mn}] = 0$$

On the basis of the above properties it can be shown (16) that the non-parametric detection tests possess the property of consistency. A detection test of H'_o against H'_1 of probability of false alarm α is said to be consistent if

$$\lim_{\substack{m \to \infty \\ n \to \infty}} \beta_{mn} = 0 \qquad (4\text{-}5)$$

where β is the probability of false dismissal. Note the dependence of β on m and n shown by the subscript mn. The property of consistency is an extremely important one since it states that for fixed z and α the decisions on the presence or absence of the signal become more reliable as more observations are obtained.

According to property (1) above the following general character of non-parametric detection statistic U_{mn} obtains when m and n are moderately large:



PROBABILITY DENSITY OF U_{mn} FOR LARGE VALUES OF m AND n, UNDER SIGNAL AND NO SIGNAL CONDITIONS

FIGURE 4.2

So then,

$$\alpha_{mn} = \left[2\pi\, \sigma_o^2[U_{mn}] \right]^{-1/2} \int_{U_\alpha}^{\infty} dy \exp\left[-1/2\, (y-E_o[U_{mn}])^2/\sigma_o^2[U_{mn}] \right] \tag{4-6}$$

or

$$\alpha_{mn} = 1/2\, (1 - \text{erf}\, \lambda\alpha)$$

where

$$\text{erf}\, x = 2(\pi)^{-1/2} \int_o^x \exp\,(-u^2)\, du \tag{4-7}$$

and

$$\lambda\alpha = \left[U_\alpha - E_o[U_{mn}] \right] / \sigma_o[U_{mn}]$$

Also,

$$\beta_{mn} = \left[2\pi\, \sigma_x^2[U_{mn}] \right]^{-1/2} \int_{\infty}^{U_\alpha} \exp\left[-1/2(y-E_z[U_{mn}])^2/\ \sigma_z^2[U_{mn}] \right] dy \tag{4-8}$$

When z is sufficiently small then using properties (3) and (4) we obtain:

$$\beta_{mn} = 1/2 \left\{ 1 - \text{erf} \left[\frac{z \left. \dfrac{dE_z[U_{mn}]}{dz} \right|_{z=o}}{2^{1/2}\, \sigma_o[U_{mn}]} - \lambda\alpha \right] \right\} \tag{4-9}$$

or

$$\frac{z \left. \dfrac{dE_z[U_{mn}]}{dz} \right|_{z=o}}{2^{1/2}\, \sigma_o[U_{mn}]} - \lambda\alpha = \text{erf}^{-1}(1 - 2\beta_{mn}) \tag{4-10}$$

and from (4-8) there follows

$$\lambda\alpha = \text{erf}^{-1}\, (1 - 2\beta_{mn}) \tag{4-11}$$

Adding Eq. (4-6) to Eq. (4-10) gives

$$\frac{z \left. \dfrac{dE_z[U_{mn}]}{dz} \right|_{z=o}}{2^{1/2}\, \sigma_o[U_{mn}]} = \text{erf}^{-1}(1-2\alpha_{mn}) + \text{erf}^{-1}\, (1-2\beta_{mn}) \tag{4-12}$$

Using property (5) the following relation obtains

$$\underline{Kz^2 \frac{mn}{m+n} = 2[\mathrm{erf}^{-1}(1-2\alpha_{mn}) + \mathrm{erf}^{-1}(1-2\beta_{mn})]^2} \tag{4-13}$$

The above relation states that

a) for decreasing signal-to-noise ratio z, the number of samples n must increase in order to maintain a constant probability of error (constant α and β). If proportional sampling is used, then an increase in the number of samples means an increase in the sampling interval and consequently a decrease in information rate.

b) for increasing signal-to-noise ratio and constant number of samples (constant information rate) the probability of error (or α, β) decreases

c) for increasing signal-to-noise ratio and constant reliability (constant probability of error) the number of samples n required decreases and thus the information rate increases.

The above relation is an extremely important one. It permits the design of a system that will guarantee a certain desired α and β for the minimum possible z. That is, if an $\alpha = 10^{-4}$, $\beta = 10^{-3}$ is desired and a signal is to be detected so weak that $z = 10^{-3}$, the only thing we need to know is K in order to determine the required samples $\frac{mn}{m+n}$. It is also possible thereby to obtain the performance characteristics of the detector in question.

To facilitate comparison between the non-likelihood detectors and the likelihood detectors the following limiting properties of the likelihood detectors are stated.[16]

The likelihood statistic U_n satisfies the following relations:

(1') U_n is asymptotically normal under H_o (no signal). The mean and standard deviation of the limiting distribution are denoted by $E_o[U]$ and $\sigma_o[U_n]$ respectively;

(2') U_n is asymptotically normal under H_1. The mean and standard deviation of this limiting distribution are given by $E_z[U_n]$ and $\sigma_z[U_n]$ respectively;

(3') $$\lim_{z \to o} \frac{\sigma_z^2 [U_n]}{\sigma_o^2 [U_n]} = 1 \tag{4-14}$$

(4') $$E_z[U_n] = E_o[U_n] + z \left. \frac{dE_z[U_n]}{dz} \right|_{z=o} + o(z^2) \tag{4-15}$$

(5') $$\lim_{z \to o} \left[\left. \frac{dE_z[U_n]}{dz} \right|_{z=o} \Big/ \sigma_o[U_n] \right]^2 = Kn \tag{4-16}$$

where K is a constant independent of n and z and dependent only on $P_o(Y)$ and $P_z(Y)$.

(6') $$\left. \frac{dE[U_n]}{dz} \right|_{z=o} \neq o \tag{4-17}$$

(7') $$\lim_{n \to \infty} \sigma_o^2 [U_n] = o \tag{4-18}$$

On the basis of the above properties, it can be shown that the likelihood tests are consistent. Also similarly to the proof for the case of non-likelihood detectors is the proof for the following property of the likelihood detectors:

$$\underline{Kz^2 n = 2[\mathrm{erf}^{-1} (1-2 \alpha_n) + \mathrm{erf}^{-1} (1-2\beta_n)]^2} \tag{4-19}$$

It was stated previously that a detection theory to be complete must also incorporate a means of comparison between different detectors. Toward this end the asymptotic relative efficiency [A.R.E.] of a non-likelihood detector $U^*_{m^*n^*}$ with respect to the non-likelihood detector U_{mn} is defined as:

$$E_{u^*,u} = \lim_{z \to 0} \frac{\frac{mn}{m+n}}{\frac{m^*n^*}{m^*+n^*}} \tag{4-20}$$

where 1) the false dismissal and the false alarm probabilities of U_{mn} and $U^*_{m^*n^*}$ are equal

$\alpha_{mn} = \alpha^*_{m^*n^*} = \alpha$

$\beta_{mn} = \beta^*_{m^*n^*} = \beta$

2) the U_{mn} and $U^*_{m^*n^*}$ detectors are for the detection of the same signal in the same noise and for the same small signal-to-noise ratio (weak signals)

For $m^* \gg n^*$ and $m \gg n$, $E_{u^*u} = \lim_{z \to 0} \frac{n}{n^*}$. Thus, the A.R.E. of one non-likelihood detector with respect to another is an indication of how many more observations one non-likelihood detector requires than the other to detect a given weak signal with a prescribed accuracy α, β when $m^* \gg n^*$ and $m \gg n$.

From the given properties (1) - (7) of the non-likelihood statistics, it can be proven that $E_{u^*u} = \frac{e(U^*_{mn})}{e(U_{mn})}$

where
$$e(U_{mn}) = \left[\left.\frac{dE_z[U_{mn}]}{dz}\right|_{z=0} \Big/ \sigma_0[U_{mn}]\right]^2 \tag{4-21}$$

$$= \frac{Kmn}{m+n}$$

$$= K\,n \text{ if } m \gg n$$

and
$$E_{u^*u} = \frac{K}{K^*}$$

It is also useful to define the A.R.E. of a non-likelihood detector $U^*_{m^*n^*}$ with respect to a likelihood detector U_n as follows:

$$E_{U^*_{m^*n^*},\ U_n} = \lim_{z \to o} \frac{n}{n^*} \tag{4-22}$$

in the direction of the same weak signal (same z) and with the same α, and β. So since

$$K^* z^2 \ \frac{n^* m^*}{m^* n^*} = 2 \left[\mathrm{erf}^{-1}(1-2\alpha) + \mathrm{erf}^{-1}(1-2\beta)\right]^2 \tag{4-23}$$

for the non-likelihood detector

and

$$Kz^2 n = 2\left[\mathrm{erf}^{-1}(1-2\alpha) + \mathrm{erf}^{-1}(1-2\beta)\right]^2 \tag{4-24}$$

for the likelihood detector

it follows that

$$E_{u^*,u} = \frac{K^*}{K} \frac{1}{1 + \frac{n^*}{m^*}} \tag{4-25}$$

Since K^* and K are independent of z, m, n, m^* and n^*, $E_{u^*,u}$ is independent of z and depends on the sample sizes only through the ratio $\frac{n^*}{m^*}$; that is the ratio of sample sizes used by the non-likelihood detector. Thus, the A.R.E. of $U^*_{m^*n^*}$ with respect to U_n is as high as possible if $m^* \gg n^*$. That is, the number of observations from the auxiliary noise source $N'(t)$ should be much larger than the number of observations from $Z(t)$.

Thus, one design criterion for the non-likelihood detector is, $m^* \gg n^*$, and so

$$E_{u^*u} = \frac{K^*}{K} \ .$$

It should be stressed that K^* and K are dependent on $P_o(Y)$ and $P_z(Y)$. The comparison of a non-likelihood detector to a likelihood detector is valid only for a particular pair of cdf's. To gain some insight on the physical significance of the asymptotic relative efficiency consider the following: one of the most important considerations in a detection problem is the length of time required to detect the signal with a certain accuracy α, β. In most cases the m^* observations obtained from $N'(t)$ by the non-parametric detector can be obtained before the n^* observations are obtained from $Z(t)$, and can be stored in the non-likelihood detector. So, the only time consumed is that used in obtaining the n^* samples from $Z(t)$. Similarly, the only time spent by the likelihood detector is that used in obtaining the n samples from $Z(t)$. If periodic sampling is employed, then n^* and n are proportional, respectively, to the time required by the non-likelihood and likelihood detectors to detect the same weak signal with the same accuracy α, β. Thus, the justification for the criterion of A.R.E. (asymptotic relative efficiency) is that for periodic sampling it gives an indication of how much better the information rate of the non-likelihood detector is than that of the likelihood detector in the detection of the same weak signal for a prescribed probability of error.

4.6 Summary of Important Properties of Non-likelihood Detectors

The following are the most significant properties of the non-likelihood detectors for their design.

1) Asymptotic normality under signal and under no-signal conditions

2) The performance relation for weak signals

$$Kz^2\, n = 2[\mathrm{erf}^{-1}(1-2\alpha_{mn}) + \mathrm{erf}^{-1}(1-2\beta_{mn})]^2 \qquad (4\text{-}26)$$

3) No knowledge of the cdf's $P_o(Y)$ and $P_z(Y)$ is required other than some functional of $P_o(Y)$ and $P_z(Y)$ e.g. max. $P_o(Y) - P_z(Y)$ for the determination of K. Note that K depends only on $P_o(Y)$ and $P_z(Y)$.

4) The efficiency of the non-likelihood detector is highest when $m \gg n$.

4.7 A Practical Design Procedure

In a practical situation a certain reliability (α, β) is specified and the weakest signal (or smallest z) to be detected is known. A case of the latter is the case of radar detection where z is a function of among others the range of the radar system. So the smallest z for the particular range can be easily determined theoretically or experimentally if the range is known. The first step in the design of a suitable detection system is to choose a non-likelihood detector from the many available e.g. a Mann-Whitney detector, or a Kolmogorov-Smirnov detector based on the Mann-Whitney and Kolmogorov-Smirnov statistical tests, respectively.

The non-likelihood statistical detection tests are asymptotically normal under signal and no-signal conditions and there the situation is as depicted in Fig. 4.2. Now, the threshold U_α that will ensure the required probability of false alarm is given by

$$\alpha = \int_{U_\alpha}^{\infty} p_o\,(U_{mn})\; d\,U_{mn} \tag{4-27}$$

where since $p_o(U_{mn})$ is Gaussian the only constants required are the mean and standard deviation of the random variable U_{mn} under no-signal conditions. These constants can be obtained experimentally, so that

$$\alpha = \frac{1}{\left[2\pi \sigma_o^2 [U_{mn}] \right]^{1/2}} \int_{U_\alpha}^{\infty} \exp \left[-1/2 \frac{(y - E_o [U_{mn}])^2}{\sigma_o^2 \; U_{mn}} \right] dy \tag{4-28}$$

writing

$$\lambda\alpha = \frac{U_\alpha - E_o[U_{mn}]}{\sigma_o[U_{mn}]}$$

and

$$\text{erf } X = 2 (\pi)^{-1/2} \int_0^X \exp (-u^2) \, du$$

it follows that

$$\alpha = 1/2 \; (1 - \text{erf } \lambda a) \tag{4-29}$$

or

$$\lambda\alpha = \text{erf}^{-1} (1-2\alpha)$$

so

$$U_\alpha = \sigma_o[U_{mn}] \, [\text{erf}^{-1} (1-2\alpha)] + E_o [U_{mn}] \tag{4-30}$$

Therefore, if $\sigma_o[U_{mn}]$ and $E_o[U_{mn}]$ are found experimentally and the required α is specified then the threshold U_α can be obtained. If U_{mn} exceeds U_α the decision that a signal is present is made. If U_{mn} is less than U_α the decision that no signal is present is made.

Having insured the required value of α through the proper selection of U_α, a value of β smaller or equal to the specified value is to be obtained. To do so, we employ the relation

$$Kz^2 \frac{mn}{m+n} = 2[\text{erf}^{-1} (1-2\alpha) + \text{erf}^{-1} (1-2\beta)]^2 \tag{4-31}$$

It is taken that $m \gg n$ to insure the highest information efficiency, so that

$$Kz^2 n = 2[\text{erf}^{-1} (1-2\alpha) + \text{erf}^{-1} (1-2\beta)]^2 \tag{4-32}$$

The right side of the equation is a known number and so is z, being the smallest signal-to-noise ratio for which a detection is to be affected. The constant K can be obtained theoretically or most often by experiment. Therefore, since everything else is known the number of observations n that will give us the specified α and a maximum β equal to the specified false dismissal probability, is obtained. Thus, the whole design problem has been completed. For the case of periodic sampling the number of observations n will give also the time required for detection and consequently the information rate.

From the above design procedure it is seen that the following quantities need to be known:

1) the constant K that depends on some functional of $P_o(Y)$ and $P_z(Y)$

2) the mean $E_o[U_{mn}]$ and $\sigma_o[U_{mn}]$ of the statistic U_{mn} under no-signal conditions.

Experimental work must be done to obtain these quantities. A detailed description of this experimental work is given in another section of this report.

4.8 General Conclusions

It was stated previously that, for a detection theory in general to be complete,

1) it must suggest the structure of the detection system

2) it must specify procedures for evaluating the performance of such systems (information rate, probability of error)

3) it must specify techniques of system comparison

In Part II of this report where particular non-likelihood detection

criteria (e.g., Mann-Whitney, Kolmogorov-Smirnov, etc.) are investigated, it is shown that the criterion itself suggests the structure of the detection system. These detection systems can be easily implemented using digital techniques.

An evaluation of the performance of the non-likelihood detectors can be made through the relation.

$$Kz^2 n = 2[\mathrm{erf}^{-1}(1-2\alpha) + \mathrm{erf}^{-1}(1-2\beta)]^2$$

Through it a lower bound for the information rate may be obtained when α, β, and the smallest z are specified. Also, when the information rate (or n) and the smallest z are specified, an upper bound for the probability of error can be had.

Using the concept of asymptotic relative efficiency a comparison can be made between the different systems. The A.R.E. for periodic sampling becomes a comparison between different systems on the basis of information rate for the same probability of error and same signal-to-noise ratio.

Thus, it is seen that the theory of non-likelihood (non-parametric) detection is complete.

Moreover, the non-likelihood detectors are the only ones appropriate for the case where little is known about the probability distributions. In fact, the only quantities that need to be known are the mean and standard deviation of U_{mn} under no-signal conditions and the constant K. These quantities are easier to obtain than the probability distributions.

Another extremely important advantage of the non-likelihood detectors is that, no assumption is required on the nature of the channel, e.g., whether the noise is additive, multiplicative or both. The only thing it

requires is that $P_o(Y)$ and $P_z(Y)$ have different means.

4.9 Experimental Work Needed

Experimental work is needed to obtain the means and standard deviations of different non-likelihood detectors under different noise densities. Also, the effect of the dependence of the observations on the performance of the system should be ascertained. An experimental set-up can be easily made to do that. Another quantity that has to be experimentally determined is the constant K that depends on $P_o(Y)$ and $P_z(Y)$ and the particular detector used. A procedure to obtain K is the following: using the same noise and signal and noise probability densities a plot of n vs $2[\mathrm{erf}^{-1}(1-2\alpha) + \mathrm{erf}^{-1}(1-2\beta)]^2$ for the same z is obtained. From the inverse of the slope of the line that is obtained, K for the detector under consideration and for the particular noise and signal used can be deduced. This experiment is repeated for different noises and signals, if the experiment is done in the laboratory, or it can be done only once in the field for the actual noise and signal that pertain to the particular communication problem of interest (ionospheric transmission, etc.).

PART II SPECIFIC NONLIKELIHOOD DETECTORS; EXAMPLES

4.10 Optimum (Suboptimum) Likelihood Detector

To facilitate comparison of the non-likelihood detectors with the likelihood detector certain results will be obtained pertaining to the likelihood detector. In particular the asymptotic relative efficiency of the likelihood detector will be obtained for various noise and signal and noise distributions.

4.11 The Optimum Detector

It is well-known that the optimum detector bases its decisions on a statistical test known as the likelihood ratio

$$L_n(y_1; \ldots; y_n; z) = \prod_{i=1}^{n} \frac{p_z(y_i)}{p_o(y_i)} \tag{4-33}$$

The important assumption that $P_z(y)$ can be expressed as a series of ascending powers of the signal-to-noise ratio z is now made. In so doing it is assumed that $P_z(y)$ has derivatives of all orders with respect to z at z = 0. It is also assumed that the series converges for all y and for all z. So,

$$P_z(y) = P_o(y) + zb(y) + 0(z^2) \qquad \begin{matrix} 0 \leq z < \infty \\ -\infty < y < \infty \end{matrix} \tag{4-34}$$

where

$$b(y) = \left. \frac{dP_z(y)}{dz} \right|_{z = o} \tag{4-35}$$

Eq. (4-34) is differentiated with respect to y to obtain

$$p_z(y) = p_o(y) + zb'(y) + 0(z^2) \qquad \begin{matrix} 0 \leq z < \infty \\ -\infty < y < \infty \end{matrix} \tag{4-36}$$

where

$$b'(y) = \frac{db(y)}{dy} = \frac{d}{dy}\left[\frac{dP_z(y)}{dz} \Big|_{z=o} \right] \tag{4-37}$$

If $P_z(y)$ is absolutely continuous, then exchanging differentiation in Eq. (4-37)

$$b'(y) = \frac{dp_z(y)}{dz} \Big|_{z=o} \tag{4-38}$$

Substituting Eq. (4-36) in Eq. (4-33) we obtain

$$L_n(y_1; \ldots; y_n; z) = 1 + z \sum_{i=1}^{n} \frac{b'(y_i)}{p_o(y_i)} + O(z)^2 \tag{4-39}$$

When a strictly increasing relationship exists between two test statistics, then these statistics are equivalent for a given detection problem. If z is sufficiently small the term $O(z)^2$ in Eq. (4-39) may be neglected. Thus, the following equivalent statistic is obtained.

$$L^*_n (y_1; \ldots; y_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{b'(y_i)}{p_o(y_i)} \tag{4-40}$$

or

$$L^*_n (y_1; \ldots; y_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial z} \ln p_z(y) \Big|_{z=o}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{b'(y_i)}{p_o(y_i)} \tag{4-41}$$

The test L^*_n is known as the locally optimum detection criterion since it is optimum only for values of z close to zero. It should be stressed that the L^*_n - test is optimum only for the particular pair of cdf's

$P_o(y)$ and $P_z(y)$ for which it has been designed. The statistic L^*_n is different for different detection problems with different cdf's $P_o(y)$ and $P_z(y)$.

It is shown in reference (16) that L^*_n satisfies properties (1') - (7') if the integral

$$\int \left[b'^2 (y) \,/\, p_o(y) \right] dz < \infty \tag{4-42}$$

is bounded. It is also shown that

$$\left.\frac{d\, E_z\, [L^*_n]}{dz}\right|_{z=o} = \int \left[b'^2(y) \,/\, p_o(y) \right] dz = K \tag{4-43}$$

The quantity

$$e[U_{mn}] = \left[\left.\frac{d\, E_z[U_{mn}]}{dz}\right|_{z=o} \Big/ \sigma_o[U_{mn}] \right]^2 \tag{4-44}$$

$$= \frac{Kmn}{m+n}$$

$$= Kn \text{ when } m >> n$$

has been named by Pitman as the efficacy of the test statistic U_{mn}.

The efficacy of L^*_n is

$$e(L^*_n) = n \int \left[b'^2 (y) \,/\, p_o(y) \right] dz \tag{4-45}$$

4.12 Detection Problems

The first problem, that of detecting a constant signal in additive normal noise, is known as the DC detection problem. The random process $N(t)$ is assumed to be a normal process. Thus, when $S(t)$ is absent the

pdf for any data element y_i $i=1, \ldots, n$ is given by

$$\phi_o(y) = (2\pi\sigma_N^2)^{-1/2} \exp\left[\left(-1/2 \frac{(y-m)^2}{\sigma_N^2}\right]\right. \tag{4-46}$$

where m and σ_N^2 are the mean and variance of the noise. The signal-to-noise ratio z for this problem is defined as

$$z = \frac{A}{\sigma_N} \tag{4-47}$$

where A is the magnitude of the constant signal.

Thus,

$$\phi_z(y) = \phi_o(y-z) \tag{4-48}$$

So for the above problem $p_o(y)$ and $p_z(y)$ are related as follows

$$p_o(y-z) = p_z(y) \tag{4-49}$$

Whenever H_o specifies a pdf $p_o(y)$ and H_1 specifies a pdf $p_z(y)$ such that Eq. (4-49) is valid, then the detection problem is known as a test for translation alternatives.

The optimum test statistic L^*_n for the DC detection problem hereon designated as t_n is shown to be

$$t_n(y_1; \ldots; y_n) = \frac{1}{n} \sum_{i=1}^{n} \frac{y_i\, \phi_o\, (y_i)}{\phi_o(y_i)} \tag{4-50}$$

$$= \frac{1}{n} \sum_{i=1}^{n} y_i$$

where

$$y = \frac{y-m}{N} \tag{4-51}$$

It is seen that t_n is independent of z and that the optimum detector is a

summing device. The efficacy of t_n is obtained as

$$e(t_n) = n \int y^2 \phi_o(y)\, dy = n \tag{4-52}$$

Thus,

$k = 1$

The second problem to be examined is the noncoherent detection of a sine-wave in additive normal narrow-band noise, hereon known as the noncoherent detection problem. The process $\{N(t)\}$ is a narrow-band normal random process with mean zero and $N(t)$ is a sample function of this process. $Y(t)$ is the same as $N(t)$ when signal is absent and is the sum of $N(t)$ and a sine-wave when signal is present. The Y_i is a random variable which is obtained from the envelope of a narrow-band normal noise when signal is absent and from the envelope of a narrow-band normal noise plus sine-wave when signal is present. The pdf of Y_i when signal is present is

$$\begin{aligned} \psi_A(y) &= \frac{y}{\sigma_N^2} \exp\left[- \frac{(y^2 + A^2)}{2\sigma_N^2} \; I_o\,(Ay/\sigma_N^2) \right] & y \geq 0 \\ &= 0 & y < 0 \end{aligned} \tag{4-53}$$

where I_o is the modified Bessel function of first kind, zero order, A is the peak of the sine-wave, and σ_N^2 is the mean square value of the noise.

The pdf when signal is absent is gotten by setting $A = 0$; thus,

$$\begin{aligned} \psi_o(y) &= \frac{y}{\sigma_N^2} \exp\left[- \frac{y^2}{2\sigma_N^2} \right] & y \geq 0 \\ &= 0 & y < 0 \end{aligned} \tag{4-54}$$

Let,

$$y = \frac{y}{\sqrt{2\sigma_N^2}} = \frac{y}{\sqrt{2\sigma_N^2}} \tag{4-55}$$

and

$$z = \frac{A^2}{2\sigma_N^2} \tag{4-56}$$

thus,

$$\begin{aligned} \psi_o(y) &= 2y \exp(-y^2) & y \geq 0 \\ &= 0 & y < 0 \end{aligned} \tag{4-57}$$

$$\begin{aligned} \psi_z(y) &= 2y \exp(-y^2 - z)\, I_o(yz^{1/2}2), & y \geq 0 \\ &= 0 & y < 0 \end{aligned} \tag{4-58}$$

The optimum test statistic for the noncoherent detection problem denoted by t_n' is

$$t_n'(y_1; \ldots; y_n) = \frac{1}{n} \sum_{i=1}^{n} (y_i^2 - 1) \tag{4-49}$$

The t_n' test is a locally most powerful test for the noncoherent detection problem since it charges for values of z other than those close to zero. The detector is a simple square-law device. The efficacy is given by

$$e(t_n') = n \int (y^2 - 1)^2 \quad \psi_o(y)\, dy = n \tag{4-60}$$

thus, $k = 1$

It can be shown(16) that in general for translation alternatives

$$e(t_n) = n \tag{4-61}$$

and

$$k = 1 \tag{4-62}$$

It should be stressed that while the likelihood detector L_n is optimum for all values of z, the modified likelihood detector L^*_n may or may not be optimum depending on the particular pair of cdf's $P_o(y)$ and $P_z(y)$.

4.13 The Mann-Whitney Detector

The Mann-Whitney test was introduced by Mann and Whitney[18] and is based on the statistic

$$V_{mn}(y_1; \ldots; y_{n+m}) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} C(y_i - y_{n+j}) \tag{4-63}$$

where

$$C(x) = 1, \text{ if } \quad x > 0$$
$$= 0, \text{ if } \quad x < 0$$

The case $x = 0$ is not considered since if $P(y)$ and $P_z(y)$ are continuous, the probability is zero that any one of the y_i's is equal to any one of the y_{n+j}'s.

The statistic V_{mn} essentially counts the number of times the magnitude of an observation y_i exceeds the magnitude of an observation y_{n+j}. This detector can be implemented using digital techniques.

Mann and Whitney have shown[18] that V_{mn} has asymptotically a normal distribution when H_o' is true if $P(y)$ is continuous and limit of $\frac{m}{n}$ exists as m, n approach infinity. Lehman[19] has shown that V_{mn} has asymptotically a normal distribution when H_1' is true if $P(y)$ and $P_z(y)$ are continuous and if the limit of $\frac{m}{n}$ exists as m, n approach infinity.

In reference (18) it is shown that

$$E_z[V_{mn}] = \int P(y)\, dP_z(y) \tag{4-64}$$

$$mn\,\sigma_z^2\, [V_{mn}] = \left(\frac{m+n+1}{12}\right) + (m-1)\,(\alpha - \epsilon_1) + (n-1)\,(\alpha - \epsilon_2) - (m+n-1)\alpha^2 \tag{4-65}$$

where

$$\alpha = \frac{1}{2} - \int = \frac{1}{2} - \int P(y)\ dP_z(y)$$

$$\epsilon_1 = \frac{1}{3} - \int P^2(y)\ dP_z(y)$$

$$\epsilon_2 = \frac{1}{3} - \int [1 - P_z(y)]^2\ dP(y)$$

when

$$z = 0$$

$$E_0\ [V_{mn}] = \frac{1}{2} \tag{4-66}$$

$$\sigma_0^2\ [V_{mn}] = \frac{(m+n+1)}{12mn} = \frac{1}{12n} \quad \text{if } m \gg n,\ m \gg 1 \tag{4-67}$$

Thus, the false-alarm probability of the Mann-Whitney detector is indeed independent of $P(y)$, since α depends only on the mean and variance of the test statistic under H_0', if the test statistic satisfies conditions (1') - (7'). It is shown in reference (16) that V_{mn} does satisfy conditions (1') - (7') if the series expansion for $P_z(y)$ and $P_z(y)$ can be performed and if the efficacy of V_{mn} is not zero.

The efficacy of V_{mn} is given by[16]

$$e(V_{mn}) = \frac{12mn}{m+n} \left[\int b'(y)\ P(y)\ dy \right]^2 = 12n \left[\int b'(y)\ P(y)\ dy \right]^2 \quad \text{if } m \gg n \tag{4-68}$$

The Mann-Whitney detector is particularly well suited to detection problems in which one of the random variables is stochastically larger than the other. Thus, the Mann-Whitney detector is very effective whenever the y_i's are stochastically larger than the y_{n+j}'s e.g., translation alternatives, noncoherent detection problems.

4.13.1 The Detection Problem of Translation Alternatives

For translation alternatives

$$p_z(y) = p(y-z) \tag{4-69}$$

where, the mean and variance of the random variable with pdf $p(y)$ are zero and one, respectively.

It should be noted here, that the asymptotic relative efficiency of any detector with respect to the likelihood detector must necessarily be less than, or at most, equal to unity. That is, any detector that has the same α and β as the likelihood detector must use a larger number of samples or it must take a longer time for it to decide.

However, if the ARE of the non-likelihood detector with respect to the modified likelihood detector L^*_n is obtained, for those cdf's for which L^*_n is not the optimum test statistic, then the ARE can be anything from zero to infinity.

For translation alternatives the efficacy of V_{mn} is[16]

$$e(V_{mn}) = \frac{12mn}{m+n}\left[\int p^2(y)\,dy\right]^2 \tag{4-70}$$

$$= 12n\left[\int p^2(y)\,dy\right]^2 \qquad \text{if } m >> n$$

Hence, the ARE of the V_{mn} detector with respect to the t_n detector for translation alternatives is, if $m >> n$[20]

$$F_{V,t}(p) = 12\left[\int p^2(y)\,dy\right]^2 \tag{4-71}$$

In particular for the DC detection problem, $p(y) = \phi_0(y)$, thus,

$$E_{V,t}(\phi_0) = 12 \left[\int (2\pi)^{-1} \exp(-y^2)\, dy \right]^2 \tag{4-72}$$

$$= 0.955!!$$

It is seen that the ARE is very high for the DC detection problem for which the t_n modified likelihood detector is optimum!!

$E_{V,t}(p)$ can be very large[16] and the minimum possible value of it is $\frac{108}{125} = 0.865$ and occurs for the density $p(y)$ given by[21]

$$p(y) = \frac{35}{100}\,(5-y^2) \qquad y^2 \leq 5 \tag{4-73}$$

$$= 0 \qquad \text{otherwise}$$

For the case of the noise having a Rayleigh distribution that is when

$$p(y) = \frac{y}{\mu^2}\, e^{-y^2/2\mu^2} \qquad \text{when signal is absent} \tag{4-74}$$

and

$$p_z(y) = \frac{y-z}{\mu^2}\, e^{\frac{-(y-z)^2}{2\mu^2}} \qquad \text{when signal is present} \tag{4-75}$$

and for

$$\mu^2 = \frac{1}{0.43} \quad \text{or} \quad \sigma_N^2 = 1$$

then

$$E_{V,t} = 12 \left[\int p^2(y)\, dy \right]^2 \tag{4-76}$$

$$= 3.48$$

Thus, the use of the Mann-Whitney detector instead of the modified likelihood detector t_n for the problem of translation alternatives does not entail a serious loss of information rate.

4.13.2 The Noncoherent Detection Problem

For the noncoherent detection problem $e(V_{mn})$ is

$$e(V_{mn}) = \frac{12mn}{m+n}\left[\int \frac{y}{2}\,\psi_0^2(y)\,dy\right]^2$$

$$= \frac{12mn}{m+n}\left[\int_0^\infty 2y^3 \exp(-2y^2)\,dy\right]^2 \tag{4-77}$$

$$= \frac{3}{4}\,\frac{mn}{m+n}$$

$$= 0.75\ n \text{ if } m \gg n$$

Thus, the ARE of the Mann-Whitney detector with respect to the t_n' modified likelihood detector is

$$E_{V,t_n'} = 0.75 \text{ for } m \gg n \tag{4-78}$$

Since the Mann-Whitney detector satisfies conditions (1') - (7') then for $z \longrightarrow 0$ (weak signals) it obeys the performance relation

$$Kz^2\ \frac{mn}{m+n} = 2\left[\mathrm{erf}^{-1}(1-2\alpha_{mn}) + \mathrm{erf}^{-1}(1-2\beta_{mn})\right]^2 \tag{4-79}$$

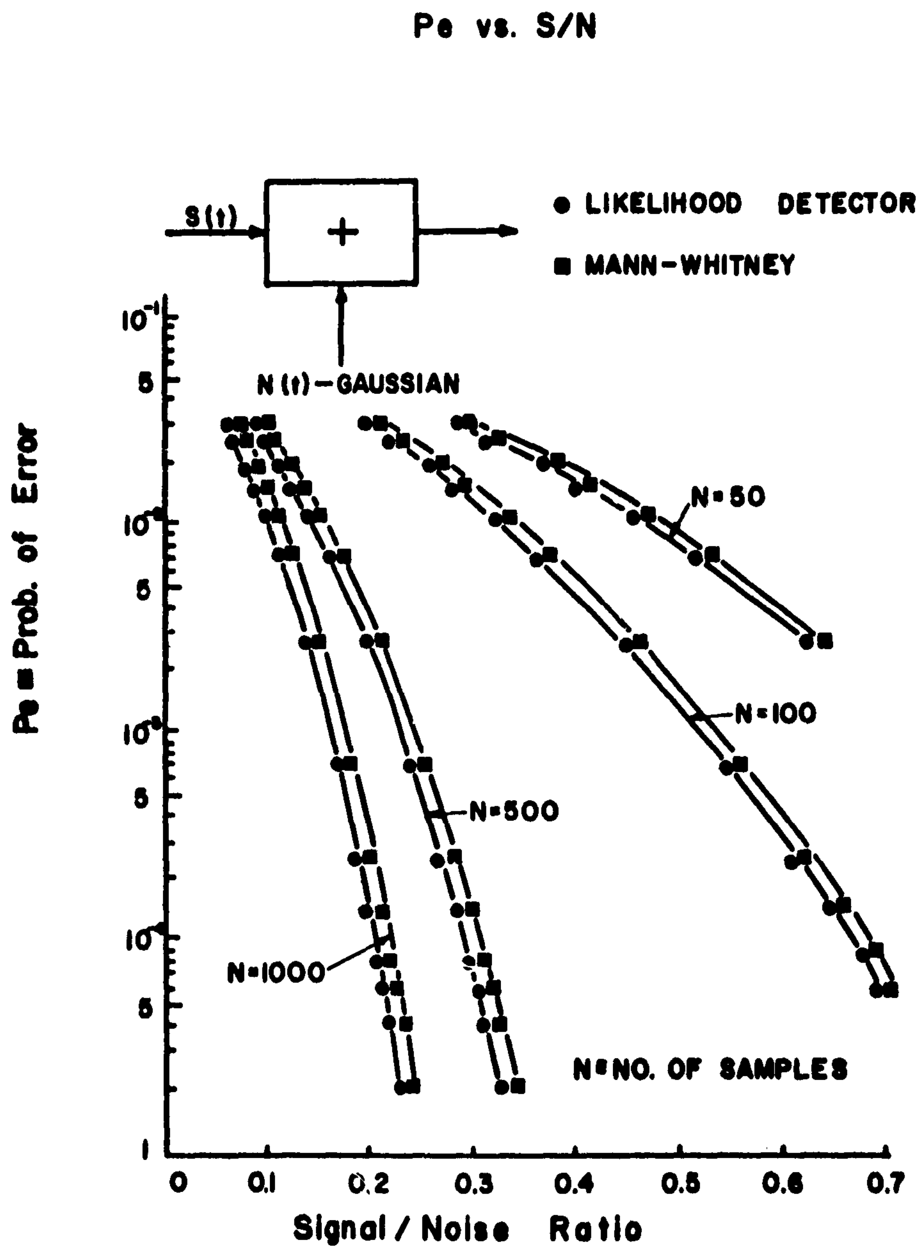
or for maximum information rate $m \gg n$ and

$$Kz^2\ n = 2\left[\mathrm{erf}^{-1}(1-2\alpha_{mn}) + \mathrm{erf}^{-1}(1-2\beta_{mn})\right]^2 \tag{4-80}$$

The above relation Eq. (4-80) has been plotted in Figs. (4.3), (4.4) and (4.5). In particular, P_e defined as

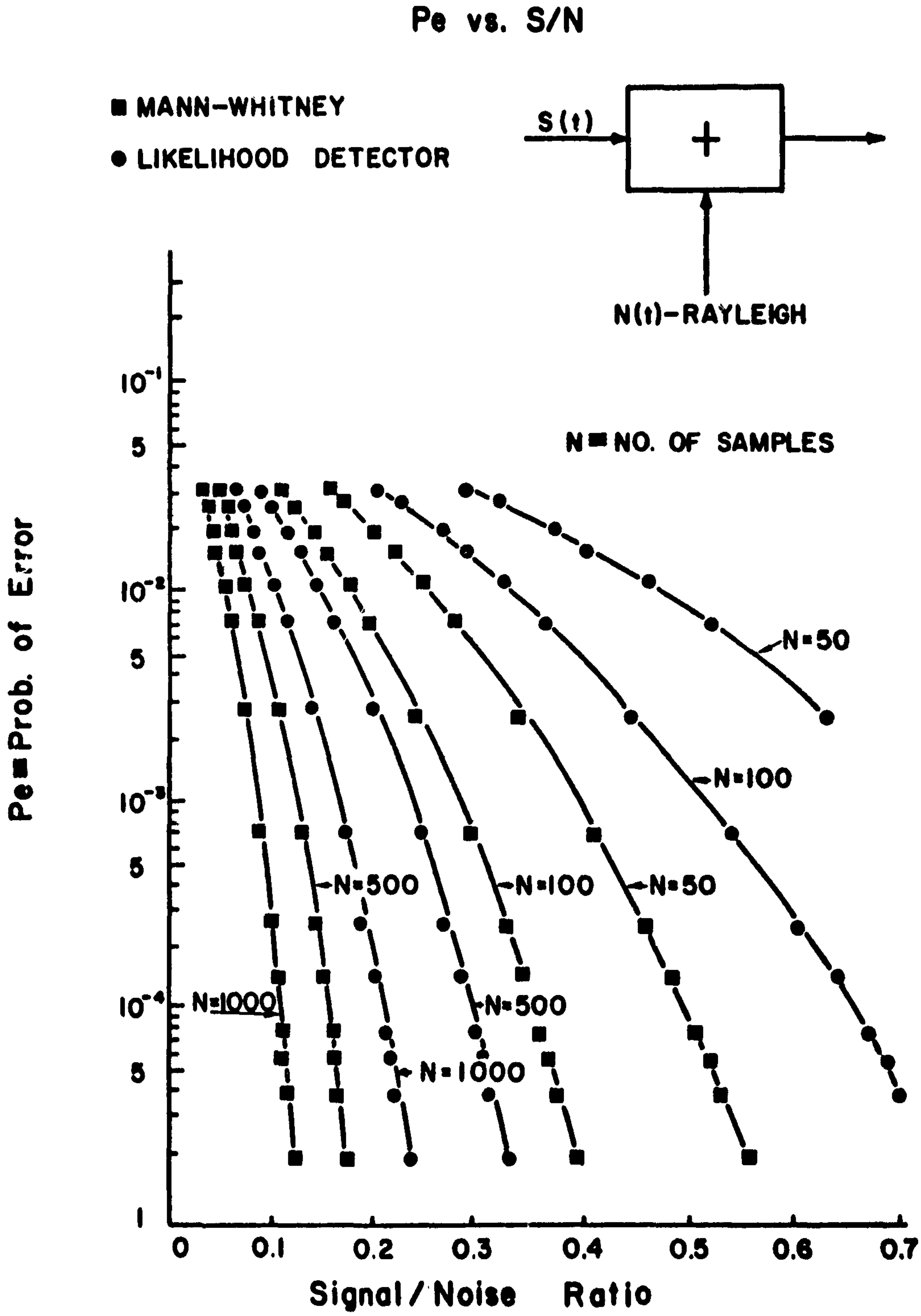
$$P_e = \alpha + \beta \tag{4-81}$$

is plotted vs. the signal-to-noise ratio z (or S/N), for various values of the number of samples (observations) n and for $m \gg n$.



PROBABILITY OF ERROR VS. SIGNAL-TO-NOISE RATIO FOR GAUSSIAN NOISE

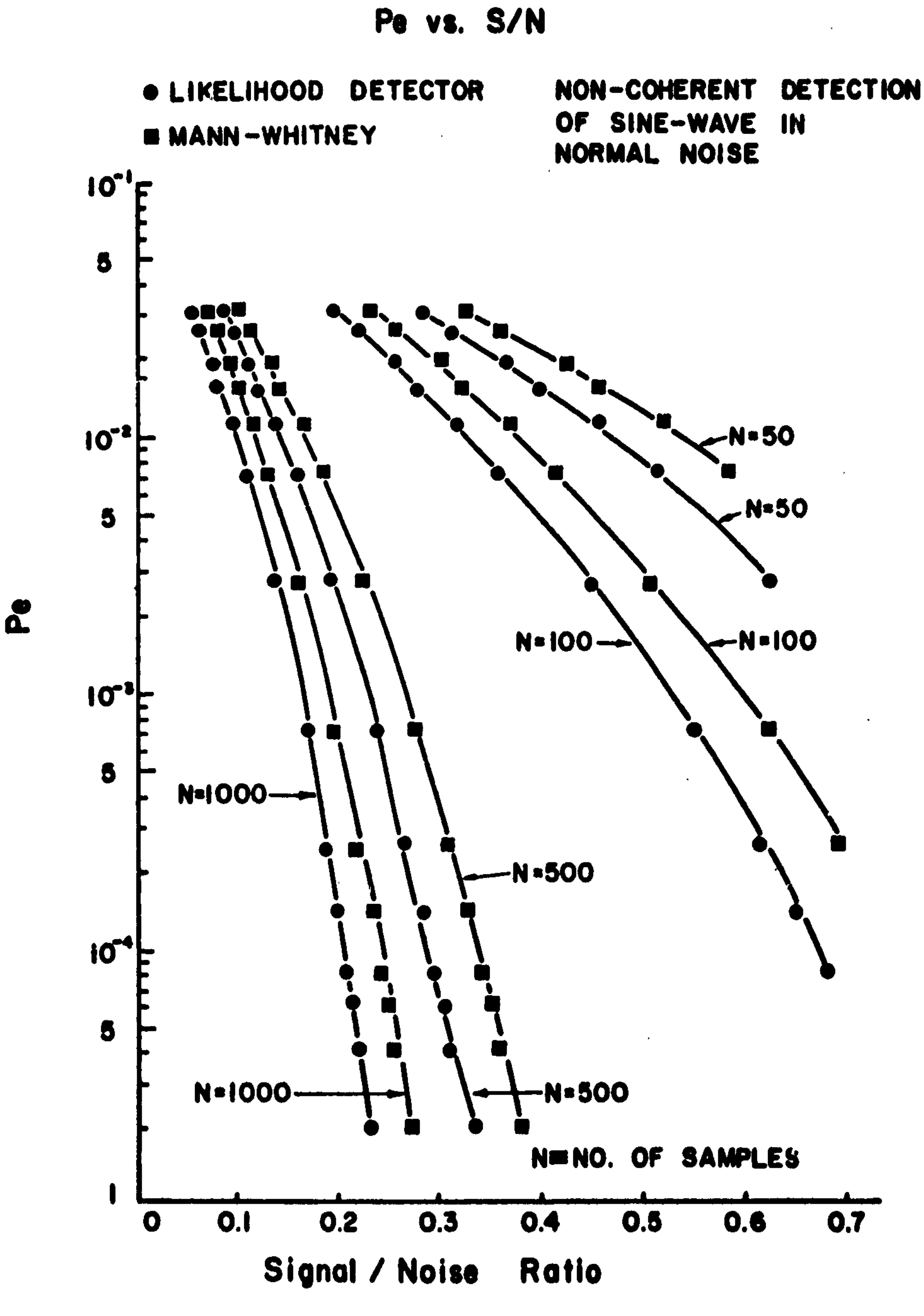
FIGURE 4.3



PROBABILITY OF ERROR VS. SIGNAL-TO-NOISE

RATIO FOR RAYLEIGH NOISE

FIGURE 4.4



PROBABILITY OF ERROR VS. SIGNAL-TO-NOISE RATIO FOR THE PROBLEM OF NON-COHERENT DETECTION OF SINE-WAVE IN GAUSSIAN NOISE

FIGURE 4.5

4.13.3 Detection of Nonstationary Signals in Noise

In all the detection problems thus far considered it was assumed that the peak signal-to-rms-noise ratio remained constant in time. In many practical situations as in scatter propagation this assumption is not justified. The noise $N(t)$ introduced in the channel is a sample function of the continuous stochastic process $\{N(t)\}$. Here it is still assumed that $\{N(t)\}$ is stationary. Thus, the cdf of y_{n+j} $j=1, \ldots, m$ is still $P(y)$.

The continuous stochastic process $\{Y(t)\}$ is not stationary when signal is present when the signal strength varies with time. Thus, the cdf of y_i $i=1, \ldots, n$ differs from the cdf of y_j $j=1, \ldots, n$ and $j \neq i$.

The detection criterion for the detection of nonstationary signals in noise (e.g. when Rayleigh fading is present in the channel) is equivalent to testing

H_0'': cdf of y_i is $P(y)$ $i = 1, \ldots, m+n$ signal is absent

against

H_1'' : cdf of y_i is $P_{z_i}(y)$, $i = 1, \ldots, n$ and the cdf of y_{n+j} is $P(y)$, $j = 1, \ldots, m$ signal is present

where some but not all of the z_i are allowed to be zero. The above hypothesis testing problem is discussed by Noether.[22]

The mean and variance of the Mann-Whitney detector for the above hypothesis is [16]

$$E_z \; [V_{mn}] = \frac{1}{n} \sum_{i=1}^{n} \int P(y) \, dP_{z_i} (y) \tag{4-82}$$

assuming the series expansion of $P_{z_i}(y)$ is possible for all $i = 1, \ldots, n$ then

$$p_{z_i}(y) = p(y) + z_i \, b' \, (y) + O(z_i)^2 \tag{4-83}$$

where

$$b'(y) = \frac{d}{dz} \; p_z(y) \Bigg|_{z = 0}$$

thus,

$$E_z \, [V_{mn}] = \frac{1}{n} \sum_{i=1}^{n} \int P(y) \;\; [p(y) + z_i \, b'(y) + 0(z_i^2)] \;\; dy \tag{4-84}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int P(y) \, p(y) \, dy + \frac{1}{n} \sum_{i=1}^{n} z_i \int b'(y) \, P(y) \, dy$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int P(y) \, dP(y) + \left\{ \frac{1}{n} \sum_{i=1}^{n} z_i \right\} \int b'(y) \, P(y) \, dy$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} + \bar{z} \int b'(y) \, P(y) \, dy$$

$$= \frac{1}{n} + \bar{z} \int b'(y) \, P(y) \, dy$$

where

$$\bar{z} = \frac{1}{n} \sum_{i=1}^{n} z_i \tag{4-85}$$

The mean and variance of the Mann-Whitney statistic remain as before, thus,

$$E_0[V_{mn}] = \frac{1}{2} \tag{4-86}$$

$$\sigma_0^2 \, [V_{mn}] = \frac{m+n}{12mn} \tag{4-87}$$

$$= \frac{1}{12n} \, , \quad \text{for } m >> n$$

and in the weak signal case when the z_i are very small, then it can be shown(16) that

$$\sigma_0^2 \, (V_{mn}) = \sigma_0^2 \, (V_{mn}) \tag{4-88}$$

It is concluded from the above that all the results obtained previously and pertaining to the Mann-Whitney detector are applicable when the signal is nonstationary (e.g., Rayleigh fading in the channel) by substituting for z the average $\bar{z}$ defined by

$$\bar{z} = \frac{1}{n} \sum_{i=1}^{n} z_i \tag{4-89}$$

Thus,

$$K\bar{z}^2 n = 2 \left[\mathrm{erf}^{-1} (1-2\alpha_{mn}) + \mathrm{erf}^{-1} (1-2\beta_{mn}) \right]^2 \quad \text{for } m >> n \tag{4-90}$$

where K has been defined as

$$K = \frac{1}{n} e(V_{mn}) \tag{4-91}$$

$$= 12 \left[\int b'(y) P(y) dy \right]^2$$

If $\int b'(y) P(y) dy$ is known, then the only information needed to obtain the sample size n in order to detect a nonstationary signal in noise with accuracy α, β is the average signal-to-noise ratio parameter $\bar{z}$. The parameters $\bar{z}$ and K may be obtained experimentally for any particular pair of signal and signal and noise distributions.

4.14 <u>The Kolmogorov-Smirnov Detector</u>

The Kolmogorov-Smirnov detector is based on the test statistic $K_{mn}(y)$ defined as

$$K_{mn}(y) = \max_{-\infty<y<\infty} \left| T_n(y) - S_m(y) \right| \tag{4-92}$$

The functions $T_n(y)$ and $S_m(y)$ are the empirical distribution functions of the samples $y_1, \ldots, y_n$, and $y_{n+1}, \ldots, y_{n+m}$, respectively, and are defined as follows

$T_n(y) = \frac{1}{n}$ number of y_i's in the sample $y_1, \ldots, y_n$ that are less or equal to y

$S_m(y) = \frac{1}{m}$ number of y_{n+j}'s in the sample $y_{n+1}, \ldots, y_{n+m}$, that are less than or equal to y

The asymptotic distribution of $K_{mn}(y)$ under H_0' was shown[23] to be

$$\text{Prob}\left[\left(\frac{mn}{m+n}\right)^{1/2} K_{mn}(y) \leq x \right] =$$

$$= 1 - 2 \sum_{j=1}^{\infty} (-1)^{j-1} \exp(-2j^2x^2) \qquad \text{if } x \geq 0 \tag{4-93}$$

$$= 0, \qquad \text{if } x < 0$$

provided P(y) is continuous and that the limit of $\frac{m}{n}$ exists as m and n approach infinity. It is noted that the limiting distribution in Eq. (4-93) is independent of the form of P(y), so that the false alarm probability of this test is independent of P(y).

The asymptotic distribution of $K_{mn}(y)$ when signal is present has not yet been investigated and in general, it is extremely difficult to obtain. However, Massey[24] has shown that an upper bound for the false dismissal probability of the K and S detector is

$$\beta_{mn} \leq (2\pi)^{-1/2} \int_{\lambda_1}^{\lambda_2} \exp\left(-\frac{x^2}{2}\right) dx \tag{4-94}$$

where

$$\lambda_1 = \frac{d - \left(\frac{m+n}{mn}\right)^{1/2} K_\alpha}{\left[\frac{P(x_o)[1-P(x_o)]}{m} + \frac{P_z(x_o)[1-P_z(x_o)]}{n}\right]^{1/2}}$$

$$\lambda_2 = \frac{d + \left(\frac{m+n}{mn}\right)^{1/2} K_\alpha}{\left[\frac{P(x_o)[1-P(x_o)]}{m} + \frac{P_z(x_o)[1-P_z(x_o)]}{n}\right]^{1/2}}$$

$$d = \max_{-\infty < x < \infty} \left| P_z(x) - P(x) \right|$$

$$= \left| P_z(x_o) - P(x_o) \right|$$

and K_α determines the critical region of false-alarm probability α and is given by

$$\text{Prob.} \left[\left(\frac{mn}{m+n}\right)^{1/2} K_{mn}(y) > K_\alpha \right] = \alpha \tag{4-95}$$

The probability distribution given in Eq. (4-93) has been published by Smirnov[25]. This table permits one to find the critical values K very easily.

The largest β occurs for λ_2 being largest and λ_1 being smallest possible. When m and n are very large then λ_2 is almost infinity. The smallest λ_1 for fixed d occurs when

$$P(x_o) = P_z(x_o) = 1/2 \tag{4-96}$$

thus,

$$\lambda_1 = 2 \left[d \left(\frac{mn}{m+n}\right)^{1/2} - K_\alpha \right] \tag{4-97}$$

So the upper bound of β is given by

$$\beta_{mn} \leq (2\pi)^{-1/2} \int_{\lambda_1}^{\lambda_2} \exp(-x^2/2)dx \tag{4-98}$$

where λ_1 is given by Eq. (4-97)

It is seen from Eq. (4-97) that λ_1 approaches infinity as m and n approach infinite. Thus,

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \beta_{mn} = 0 \tag{4-99}$$

which means that the Kolmogorov-Smirnov detector possesses the important property of consistency. Note that this is true for all continuous cdf's $P(y)$ and $P_z(y)$.

The statistic K_{mn} does not satisfy condition (1) so it is not possible to use the methods developed in Part I to obtain the asymptotic relative efficiency. However, one may proceed as follows. The relation between K_α, and α is given by Eq. (4-93)

$$2 \sum_{j=1}^{\infty} (-1)^{j-1} \exp(-2j^2 K^2) = \alpha \tag{4-100}$$

When α is small then K_α is large so that in the series expansion of Eq. (4-100) the only significant term is that for $j = 1$.

Thus,

$$K = \left[\frac{1}{2} \quad \ln \frac{2}{\alpha} \right]^{1/2} \tag{4-101}$$

The upper bound for the false dismissal probability β is

$$\beta \leq (2\pi)^{-1/2} \int_{\lambda_1}^{\infty} \exp\left(-x^2/2\right) dx \tag{4-102}$$

or

$$\beta \leq \frac{1}{2} \left\{ 1 - \operatorname{erf}\left(\frac{\lambda 1}{\sqrt{2}}\right) \right\} = \frac{1}{2} \left\{ 1 - \operatorname{erf}\left[\frac{\lambda\, 1}{(2)^{1/2}} \right] \right\}$$

which because of Eq. (4-97) becomes

$$d \left[\frac{mn}{m + n} \right]^{1/2} - K_\alpha \leq \frac{1}{(2)^{1/2}} \operatorname{erf}^{-1} (1-2\beta) \tag{4-103}$$

adding Eqs. (4-101) and (4-103) yields

$$d^2 \frac{mn}{m+n} \leq \left\{ \left[\frac{1}{2} \quad \ln \frac{2}{\alpha} \right]^{1/2} + \frac{1}{(2)^{1/2}} \operatorname{erf}^{-1}(1-2\beta) \right\}^2 \tag{4-104}$$

The quantity d can be obtained for translation alternatives and for very small z, (weak signals), as

$$d = \max_{-\infty < y < \infty} \left| P_z(y) - P(y) \right| \tag{4-105}$$

$$= \max_{-\infty < y < \infty} \left| P(y-z) - P(y) \right|$$

$$= \max_{-\infty < y < \infty} \lim_{z \to 0} \left\{ z \left[\frac{P(y-z) - P(y)}{z} \right] \right\}$$

Assuming that

$$\lim_{z \to 0} \frac{P(y-z) - P(y)}{z} = P(y) \tag{4-106}$$

exists for all y, then

$$d = z \max_{\substack{-\infty < y < \infty \\ \text{translation} \\ \text{alternatives}}} p(y) \tag{4-107}$$

$$= z\, M_f$$

where

$$M_f = \max_{-\infty < y < \infty} p(y)$$

Thus, Eq. (4-104) becomes

$$z^2 M_f^{\,2} \frac{mn}{m+n} \leq \left\{ \left(\frac{1}{2} \ell n \frac{2}{\alpha}\right)^{1/2} + \left(\frac{1}{2}\right)^{1/2} \operatorname{erf}^{-1} (1-2\beta) \right\}^2 \tag{4-108}$$

For the likelihood detector it was found that

$$e(t_n) = n \qquad \text{for translation alternatives} \tag{4-109}$$

or K = 1

thus for the t_n - test and for translation alternatives the following relations obtain

$$z^2 n = 2 \left[\operatorname{erf}^{-1} (1-2\alpha) + \operatorname{erf}^{-1} (1-2\beta) \right]^2 \tag{4-110}$$

In Part I, it was defined that

$$E_{k',t} = \lim_{z \to 0} \frac{n}{n^*} \tag{4-111}$$

where both tests are for the same value of z and accuracy α, β. Thus, a lower bound for $E_{k,t}$ may be obtained as follows

$$E_{k,t} \geq 4M_f^2 \; Q(\alpha,\beta) \; \frac{m^*}{m^*+n^*} \tag{4-112}$$

where m*, n* are the number of samples for the Kolmogorov-Smirnov detector, and

$$Q(\alpha,\beta) = \frac{[\mathrm{erf}^{-1}(1-2\alpha) + \mathrm{erf}^{-1}(1-2\beta)]^2}{[(\ell n \frac{2}{\alpha})^{1/2} + \mathrm{erf}^{-1}(1-2\beta)]^2}$$

$E_{k,t}$ is as large as possible when $m^* >> n^*$ and then

$$E_{k,t} \geq 4M_f^2 \; Q(\alpha,\beta) \qquad m >> n \tag{4-113}$$

for translation alternatives

Thus, for the various problems discussed before it follows that

4.14.1 DC Detection Problem

$$p(y) = \phi_0(y) \text{ and } M_{\phi_0} = (2\pi)^{-1/2} \tag{4-114}$$

so

$$E_{k,t}(\phi_0) \geq \frac{2}{\pi} Q(\alpha,\beta) = 0.64\, Q(\alpha,\beta) \tag{4-115}$$

For a value of $\alpha = \beta = 10^{-3}$

$$E_{k,t}(\phi_0) \geq 0.50$$

and for $\alpha = \beta = 10^{-5}$

$$E_{k,t}(\phi_0) \geq 0.55$$

The above values are sufficiently high to warrant use of the Kolmogorov-Smirnov detector whenever this detector is appropriate.

4.14.2 Translation Alternatives

It is shown in reference (16) that a lower bound for $E_{k,t}$ for translation alternatives exists, and it is

$$E_{k,t} \geq \frac{1}{3} Q(\alpha,\beta) \tag{4-116}$$

which for $\alpha = \beta = 10^{-3}$ becomes

$$E_{k,t} \geq 0.26$$

4.14.3 Rayleigh Noise Detection Problem

For this problem one obtains

$$4M_f^2 = 9.28 \tag{4-117}$$

and, therefore,

$$E_{k,t}(\text{Rayleigh}) \geq 9.28\, Q(\alpha,\beta) \tag{4-118}$$

which for $\alpha = \beta = 10^{-3}$ becomes

$$E_{k,t}(\text{Rayleigh}) \geq 7.3$$

4.14.4 Noncoherent Detection Problem

It can be shown easily for[16] the noncoherent detection problem that

$$E_{k,t} \geq \left(\frac{2}{e}\right)^2 Q(\alpha,\beta) \qquad m >> n \tag{4-119}$$

which for $\alpha = \beta = 10^{-3}$ becomes

$$E_{k,t} \geq 0.42$$

and for $\alpha = \beta = 10^{-5}$ becomes

$$E_{k,t} \geq 0.47$$

4.15 Rank Detectors

The rank detectors to be discussed in this section are optimum in the sense that for a given α, m, and n they have the smallest β among all size -α rank tests. It should be stressed that these detectors are optimum only for a particular pair of cdf's $P(y)$ and $P_z(y)$.

It has been shown[26] that if $p_z(y)$ is greater than zero whenever $p(y)$ is greater than zero, then the optimum rank detector of H'_0 against H'_1 is based on the statistic

$$R_{mn}(y_{N1}; \ldots ; y_{NN}: z) =$$

$$= E_0 \left[\prod_{i=1}^{N} p_z\, y_{iy_{Ni}} / p(z_{iy_{Ni}}) \right] \tag{4-120}$$

where y_i is the i-th smallest of the combined sample $y_1, \ldots, y_{n+m}$ and y_{Ni} is defined as

$$y_{Ni} = 1, \text{ if } y_i \text{ falls in } y_1, \ldots, y_n$$
$$= 0, \text{ if } y_i \text{ falls in } y_{n+1}, \ldots, y_{n+m}$$

where $N = n+m$

For weak signals, z is very small and substituting the series expansion for $P_z(y)$ in Eq. (4-120) we obtain an equivalent expression

$$R^*_{mn}\,(y_{N1}; \ldots : y_{NN} = \frac{1}{n} \sum_{i=1}^{N} \alpha_{Ni}\, y_{Ni} \tag{4-121}$$

Where

$$\alpha_{Ni} = E_0 \left[b'\,(y_i)/p(y_i) \right]$$

In order to use R^*_{mn} we must know the numbers α_{Ni}. These are very difficult to compute. The function $b'(x)/p(x)$ is found from the particular pair of cdf's $P(y)$ and $P_z(y)$ for which the rank detector is optimum. The complexity of the function $b'(x)/p(x)$, and of the cdf $P(y)$ determine whether it is feasible to obtain the numbers α_{Ni}.

It can be shown[16] that R^*_{mn} satisfies conditions (1') - (7') if the integral

$$\int \left[b'^2(y)/p(y) \right] dy$$

is bounded and if $b'(y)$ is not identically zero.

The mean and variance of R^*_{mn} are[16]

$$E_0(R^*_{mn}) = \int b'(y)\, dy = 0 \tag{4-122}$$

$$\sigma_0^2(R^*_{mn}) = \frac{m}{n\,N} \int \left[b'^2(y)/p(y) \right] dy \tag{4-123}$$

Also the efficacy of R^*_{mn} can be shown[16] to be

$$e(R^*_{mn}) = \frac{mn}{N} \int \frac{b'^2(y)}{p(y)}\, dy$$

$$= n \int \left[b'^2(y)/p(y) \right] dy \qquad m >> n \tag{4-124}$$

The asymptotic relative efficiency of R^*_{mn} with respect to the likelihood detector L^*_n was proven[16] to be

$$E_{R^*,L^*} = 1 \quad \text{for } m >> n \tag{4-125}$$

Eq. (4-125) above states the extremely important fact that the rank detector based on R^*_{mn} has the same information efficiency as the likelihood detector based on L^*_n, when the efficiencies are calculated for the particular pair of cdf's $P(y)$ and $P_z(y)$ for which both detectors are optimum. Moreover, the rank detection has the additional advantage that its false alarm probability does not depend on the actual cdf of the y_i's under no signal conditions.

4.15.1 DC Detection Problem

For this detection problem the statistic R^*_{mn} takes the form

$$R^*_{mn}\,(y_{N1};\ \dots : y_{NN}) = \frac{1}{n} \sum_{i=1}^{N} E_0\ [y_i]\ \ y_{Ni} \tag{4-126}$$

where $E_0(y_i)$ is the expected value of the i-th smallest observation of a sample of N from the standard normal distribution.

It has been shown[16] that $E_{R^*_{mn}, t}$ is always greater or equal to one and equals one only if p(y) is the standard normal density. Thus, it is always more efficient to use R^*_{mn} than the likelihood detector based on t_n, for the problem of translation alternatives.

4.15.2 Translation Alternatives

It can be shown[16] that an upper bound for the sample n exists, and it is

$$n \leq \frac{2}{z^2} \left[\mathrm{erf}^{-1} (1-\alpha_{mn}) + \mathrm{erf}^{-1} (1-2\beta_{mn}) \right]^2 \qquad m >> n \tag{4-127}$$

for translation alternatives and for R^*_{mn} as given by Eq. (4-126).

4.15.3 Noncoherent Detection Problem

For this problem an equivalent statistic for R^*_{mn} is T_{mn} defined as

$$T_{mn} (y_{N1}; \ldots : y_{NN}) = \frac{1}{n} \sum_{i=1}^{N} y_{Ni} \sum_{j=N+1-i}^{N} j^{-1} \tag{4-128}$$

According to Eq. (4-125) the nonlikelihood detector based on T_{mn} has the same information efficiency as the likelihood detector based on t_n'. In addition, the rank detector based on T_{mn} has the decided advantage that its false alarm probability is independent of P(y).

CHAPTER V

OPTIMIZATION OF SIGNALING WAVEFORMS

5.1 Introduction

In communication systems, the transmitted signal seems to be that part which has until now received the least scrutiny in the light of modern communication theory. Instead, most communication system analysis usually begins by taking for granted one of the conventional modulations, or a choice of signals is made from a number of traditional types, on the basis of past experience.

Actually, all other factors being fixed, a suitably designed signal holds the promise of transferring to the transmitter some of the signal processing operations now called for at the receiver in order to achieve near-optimum reception. This would be of particular interest in ground-to-air and ground-to-space communication. Aside from this, an improvement in performance (error rate) of any given system is indicated if the transmitted signal is optimized with respect to the characteristics of the channel.

In order to determine the extent of possible improvements and to examine some of the problems involved in effecting such improvements, this investigation of Signal Design was initiated, and the work performed in this area thus far is reported in this chapter.

First comes a discussion of the signal design problems which arise in digital communication systems, with a breakdown into various categories, according to the constraints imposed by the system requirements and the

channel. Then follows a discussion of the specific problem investigated so far and the results obtained.

The work so far has been concerned with the determination of optimum waveshapes which will not give rise to intersymbol interference in a dispersive channel -- i.e., a "channel with memory" -- if the channel characteristics are assumed known. This differs from other published work in signal design, as indicated in section 5.2. A very simple channel model is considered in order that specific results may be discussed.

In section 5.3, reference is made to recent literature on waveforms which eliminate intersymbol interference, and several simple examples of such waveforms are presented. If the channel, transmission rate, and transmitted energy (per waveform) are specified, many such waveforms can be found, but they will generally result in different values of received energy. Therefore, in section 5.4, those waveforms are found which maximize the received energy, given a certain channel. Such waveforms are optimum if the receiver contains a matched filter.

The elimination of intersymbol interference is accomplished at the expense of signal energy. This trade-off is examined in section 5.5. How accurately must the channel parameters be known in order to make possible near-optimum performance? This question is investigated in section 5.6. In section 5.7 it is shown that further optimization is possible if the transmitted waveforms are permitted to overlap somewhat.

Although the results obtained thus far are very interesting, it is clear that considerably more work is required to illuminate the problem considered here as well as other applications of optimum signal design.

5.2 Outline of Problems

5.2.1 General Discussion

One of the problems involved in the design of a communication system is the specification of waveforms to be transmitted. It is a difficult problem for the following reasons:

1) Often the most important factor determining the optimum transmission waveform is the exact nature of the transmission channel, which, in thecase of radio communication, is usually only vaguely known, and in general also varies considerably with time.

2) All portions of the system impose requirements -- some conflicting -- on the signal waveform, making the optimization of the signal waveform often a difficult, if not impossible analytical problem; also, a mathematical solution, if successful, may still not be very useful if it results in a waveshape that is difficult to generate.

Of recent interest are feedback communication systems. These could be arranged to measure channel parameters -- continuously, if necessary -- and then to use the channel estimates thus obtained at the transmitter for proper signal shaping. This is a way of overcoming the difficulty no. 1) above.

The purpose of the current study is to investigate the maximum improvements which can be obtained by proper signal design and thus pertains to item no. 2) above. For this reason in all subsequent discussion it will be assumed that the transmission channel is completely specified.

It should also be pointed out that the scope of this program is restricted to digital communication systems. The situation under consideration may thus be represented by the simple diagram in Fig. 5.1.

COMMUNICATION SYSTEM, AS CONSIDERED IN THIS CHAPTER

FIGURE 5.1

The waveform generator produces a train of waveforms $e_i(t)$ which are selected from a signal alphabet. The channel is completely specified; it may contain sources (noise, interference), delays, and non-linearities. The "waveform observer" is a suitable device for deciding on the transmitted symbol from the observed waveform, $e_o(t)$. Note that filters and other networks following the waveform generator and preceding the waveform observer can be conveniently lumped into the channel.

5.2.2 Factors Which Determine the Transmitted Waveforms

In any given communication system, a number of requirements restrict the types of signals to be considered for $e_i(t)$. Often the requirements combine to limit $e_i(t)$ to only a single pair of (binary) waveforms. These requirements can be grouped for convenience into three basic categories:

A) The exact nature of the channel

B) The performance criterion

C) Specified constraints concerning the transmission and reception processes.

Typical examples of each category follow.

5.2.2.1 The Channel

Various ways in which a channel may act on a signal are:

a) Dispersion, representable by transmission through a lumped constant network

b) Dispersion, representable by transmission through a distributed constant network

c) Multipath

d) Nonlinear operation, as in the case of Doppler

e) Some combination of the above

These effects are present to various extents, regardless of the noise; so that in conjunction with any of the above cases might be considered

a) No appreciable noise or interference present

b) Noise of specified statistics present

c) Specified interfering signals present

5.2.2.2 The Performance Criterion

This must be determined in the case of any communication system design and depends on the nature and purpose of the system. Sometimes several criteria are to be satisfied.

Examples of such criteria are:

a) Minimization of intersymbol interference

b) Minimization of adjacent channel interference

c) Minimization of error rate

d) Minimization of cost, if suitably defined

5.2.2.3 Constraints

These are additional requirements for the communication system which are initially specified. They may be:

a) Alphabet size. A binary, ternary, or larger signal alphabet may be specified.

b) Signaling rate. A certain fixed rate may be specified.

c) Restrictions regarding the generation of waveforms may be given, such as maximum bandwidth, maximum average signal power, maximum peak power.

d) The detection system may be specified as coherent, or incoherent; maximum permissible delay or storage capacity at the receiver may be specified.

e) The maximum allowable degradation in system performance resulting from specified changes in certain system parameters.

5.2.3 Problems Investigated in the Past

Considerable work has already been done for some of these cases by a number of investigators.

Optimum signals to be used in the presence of white and colored Gaussian noise have been determined for channels representable by linear constant parameter networks for the case where the duration of each signaling element is substantially larger than the significant part of the channel impulse response, as discussed by Middleton (Ref. 27, Chapter 23) and Lerner (Ref. 28, Chapters 8 and 11). Signals suitable for use with multipath channels have been found to be maximal length binary shift register sequences, as discussed by Price and Green (Ref. 29). Transmission with Doppler has primarily been investigated in connection with Radar (Refs. 30 and 31) which gives rise to different requirements than a communication link because the pertinent information in the received radar signal is its delay.

5.2.4 Specific Problem Considered in this Chapter

Discussion henceforth is limited to channels representable by linear lumped constant networks, with additive Gaussian noise of constant spectral density. No restriction on signaling rate is imposed. The criterion a), minimization of intersymbol interference is applied first. This is then combined with criterion c), minimization of error rate, which in the presence of interfering white Gaussian noise implied maximum energy transfer through the channel. An arbitrary fixed signaling rate is assumed.

5.3 Complete Elimination of Intersymbol Interference

It has been shown by Gerst and Diamond (Ref. 32) that in the case of pulse transmission through linear lumped constant networks, intersymbol interference can be completely eliminated by the use of appropriate signaling waveforms. They also show how to find such waveforms, given the transfer function of the network under consideration. Section 5.3.1 is a summary of the results obtained by Gerst and Diamond which are pertinent to the problem under consideration. The word "pulse" is used in the following and subsequent sections to mean a waveform which is non-zero only in a specified finite time interval.

5.3.1 Waveforms which Achieve Complete Elimination of Intersymbol Interference

a) For any lumped-element constant parameter network, there exist input pulses of arbitrary length a, such that the corresponding outputs of the system are pulses of the same length a.

b) Pulses which satisfy a) may be constructed by one of the following methods:

Method I:

The Laplace-transform $E_i(s)$ of the desired input pulse of duration a is given by

$$E_i(s) = G(s) \cdot \prod_{j=1}^{n} \left\{1 - \exp\left[-\frac{a - \max(a_i)}{n}(s - \gamma_j)\right]\right\} ; \tag{5-1}$$

where $G(s)$ is an entire function* of the form $\frac{1}{D(s)} \sum_{i=1}^{k} \epsilon^{-a_i s} P_i(s)$,

$P_i(s)$, $i = 1, \ldots, k$, and

$D(s)$ being polynomials in s, with the $P_i(s)$ of lower degree than $D(s)$,

a_i, $i = 1, \ldots, k$ are non-negative real numbers smaller than a, and

γ_j, $j = 1, \ldots, n$ are the n poles of the network transfer function.

The simplest function satisfying the requirements for $G(s)$ is, therefore,

$$G(s) = \frac{1 - \epsilon^{-\frac{a}{n+1}s}}{s} . \tag{5-2}$$

Method II:

If $H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=o}^{m} k_i s^i}{\sum_{i=o}^{n} h_i s^i}$ is the transfer function of the network where $m \leq n$ and $h_n = 1$,

then we have for the input pulse, $e_i(t)$, and the associated output pulse $e_o(t)$:

---

* An entire function is a function of a complex variable which is analytic and has no singularities in the finite plane.

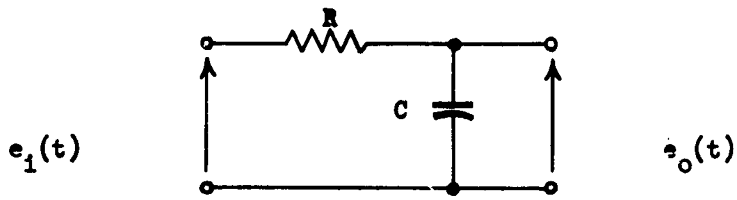
$$\left.\begin{aligned} e_i(t) &= h_o e_1(t) + h_1 e_1'(t) + \ldots\ldots\ldots + h_n e_1^{(n)}(t), \\ e_o(t) &= k_o e_1(t) + k_1 e_1'(t) + \ldots\ldots\ldots + k_m e_1^{(m)}(t); \end{aligned}\right\} \tag{5-3}$$

where $e_1(t)$ is a pulse which has the specified duration and is differentiable n times.

5.3.2 Specific Examples

5.3.2.1 RC Low-pass Network

a) Given the following network, which has the transfer function $H(s) = \frac{\alpha}{s+\alpha}$, $\alpha = \frac{1}{RC}$:



RC LOW-PASS NETWORK

FIGURE 5.2

By method I of section 5.3.1, using

$$E_i(s) = \frac{1-\epsilon^{-\frac{1}{2}as}}{s}\left[1-\epsilon^{-\frac{1}{2}a(s+\alpha)}\right], \tag{5-4}$$

the following input and output functions are obtained, where u(t) is the unit step:

$$e_i(t) = u(t)-(1-\epsilon^{-\frac{1}{2}\alpha a})\, u(t-\tfrac{a}{2}) + \epsilon^{-\frac{1}{2}\alpha a}\, u(t-a)\ ; \tag{5-5}$$

$$\begin{aligned} e_o(t) = (1-\epsilon^{-\alpha t})\, u(t)-(1-\epsilon^{-\frac{1}{2}\alpha a})(1-\epsilon^{-\alpha(t-\frac{1}{2}a)})u(t-\tfrac{a}{2}) \\ + e^{-\frac{1}{2}\alpha a}(1-\epsilon^{-\alpha(t-a)})\, u(t-a)\ . \end{aligned} \tag{5-6}$$

Sketches of these functions for typical values of $a\alpha$ are shown in Fig. 5.3.

b) If, instead, $E_1(s)$ is chosen to be

$$E_1(s) = \left(\frac{1-\epsilon^{-as/3}}{s}\right)^2 \left(1-\epsilon^{-(a/3)(s+\alpha)}\right), \tag{5-7}$$

then a typical pair of input-output waveforms is the one shown in Fig. 5.4 for the case $a\alpha = 1$. The input pulse is now continuous.

c) Applying method II to the RC low-pass network, one notes that the input and output waveforms are of the form

$$\left.\begin{aligned} e_i(t) &= \alpha e_1(t) + e_1'(t)\,, \\ e_o(t) &= \alpha e_1(t)\,; \end{aligned}\right\} \tag{5-8}$$

where $e_1(t)$ is a pulse waveform which must be a differentiable function of time.

A suitable function to be used for $e_1(t)$ is

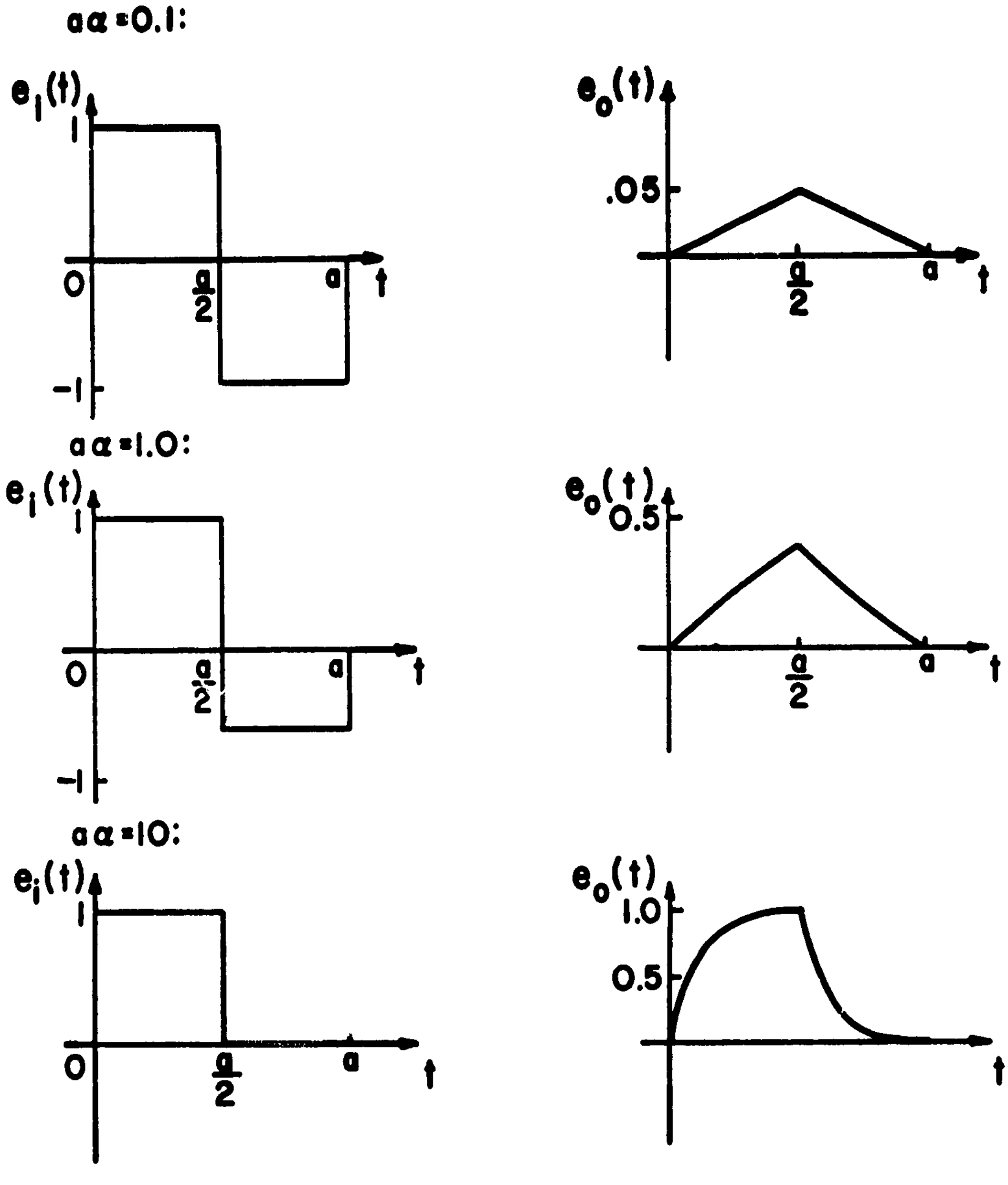
$$e_1(t) = \begin{cases} \left(1-\cos\frac{2\pi t}{a}\right)^2\,, & 0 \leq t \leq a \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{5-9}$$

With this choice for $e_1(t)$, the following input and output functions result:

$$e_i(t) = \alpha\left(1-\cos\frac{2\pi t}{a}\right)^2 - \frac{2\pi}{a}\left(\sin\frac{4\pi t}{a} - 2\sin\frac{2\pi t}{a}\right), \quad 0 \leq t \leq a \tag{5-10}$$

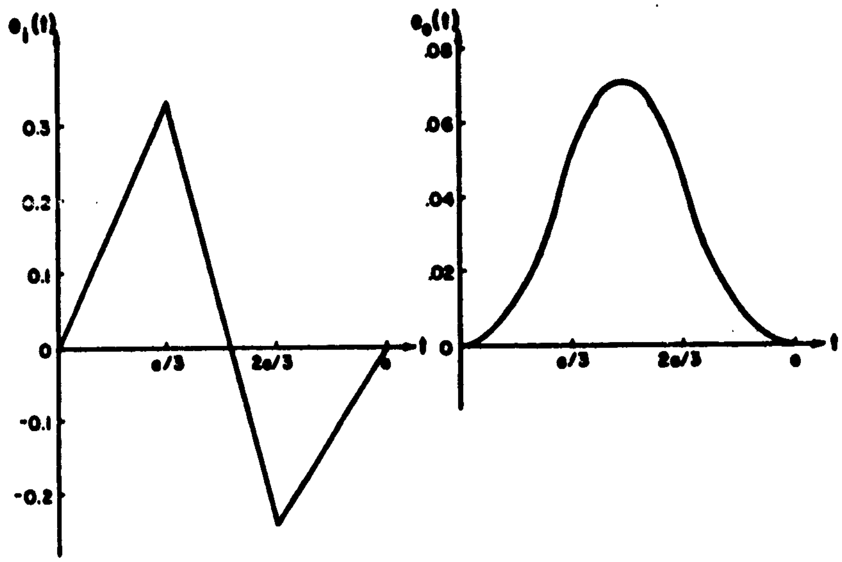
$$e_o(t) = \alpha\left(1-\cos\frac{2\pi t}{a}\right)^2\,, \quad 0 \leq t \leq a \tag{5-11}$$

For several values of $a\alpha$, the waveforms are shown in Fig. 5.5.



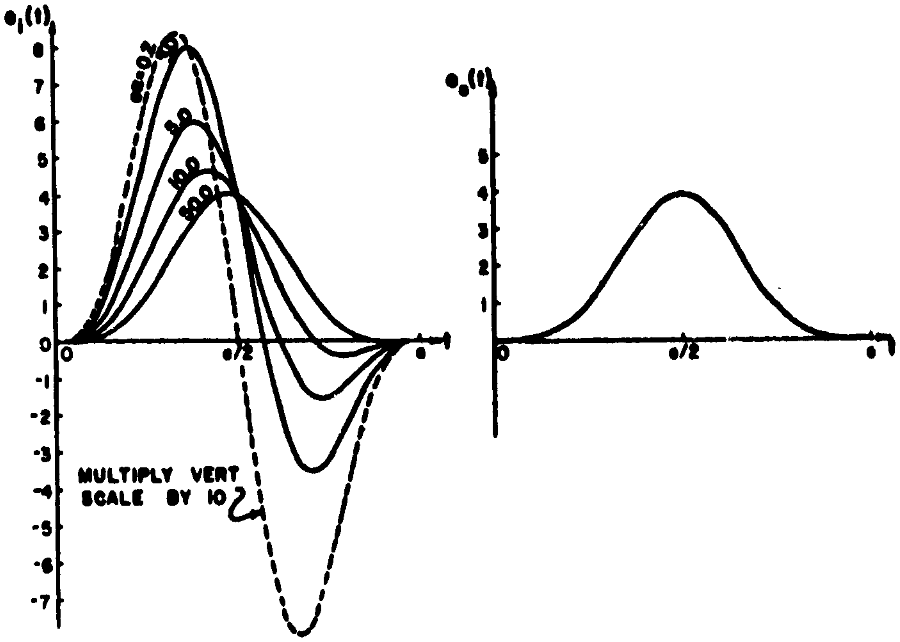
INPUT-OUTPUT PULSE PAIRS OBTAINED IN SECTION 5.3.2.1(a)

FIGURE 5.3



INPUT-OUTPUT PULSE PAIR OBTAINED IN SECTION 5.3.2.1(b)

FIGURE 5.4

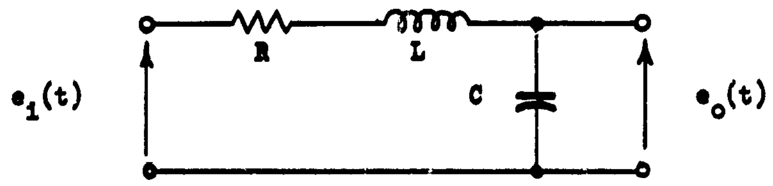


INPUT-OUTPUT PULSE PAIR OBTAINED IN SECTION 5.3.2.1 (c)

FIGURE 5.5

5.3.2.2 RLC Low-pass Network

a) The RLC network shown below is considered next. To be specific, $R = \sqrt{\frac{2L}{C}}$ is assumed.*



RLC LOW-PASS NETWORK

FIGURE 5.6

The transfer function is $H(s) = \frac{2\alpha^2}{s^2+2s\alpha+2\alpha^2}$, where $\alpha = \frac{R}{2L} = \frac{1}{\sqrt{2LC}}$.

By method I, using

$$E_i(s) = \frac{1-\epsilon^{-\frac{as}{3}}}{s}\left(1-\epsilon^{-(a/3)[s+\alpha(1+j)]}\right)\left(1-\epsilon^{-(a/3)[s+\alpha(1-j)]}\right) , \tag{5-12}$$

the input and output pulses are:

$$e_i(t) = u(t)-(1+2\epsilon^{-a\alpha/3}\cos\frac{a\alpha}{3})u(t-\frac{a}{3})+(\epsilon^{-2a\alpha/3}+2\epsilon^{-a\alpha/3}\cos\frac{a\alpha}{3})u(t-\frac{2a}{3})$$

$$- \epsilon^{-2a\alpha/3}\, u(t-a) , \tag{5-13}$$

$$e_o(t) = h_{-1}(t) - (1+2\epsilon^{-a\alpha/3}\cos\frac{a\alpha}{3})\, h_{-1}(t-\frac{a}{3}) +$$

$$(e^{-2a\alpha/3}+2\epsilon^{-a\alpha/3}\cos\frac{a\alpha}{3})h_{-1}(t-\frac{2a}{3}) -\epsilon^{-2a\alpha/3}h_{-1}(t-a), \tag{5-14}$$

---

* This example is also presented in Ref. 32.

where

$$h_{-1}(t) = u(t)[1-\epsilon^{-\alpha t}(\cos\ \alpha t + \sin \alpha t)] \tag{5-15}$$

Typical waveforms are shown in Fig. 5.7.

b) Method II is now applied to the same network, and the same auxiliary function $e_1(t)$ is chosen as was used in section 5.3.2.1. The resulting input and output functions are:

$$e_i(t) = (2\alpha^2)(1-\cos\frac{2\pi t}{a})^2 + 8\frac{\pi\alpha}{a}(\sin\frac{2\pi t}{a} - \frac{1}{2}\sin\frac{4\pi t}{a}) + 8\frac{\pi^2}{a^2}(\cos\frac{2\pi t}{a} - \cos\frac{4\pi t}{a})\ , \quad 0 \le t \le a. \tag{5-16}$$

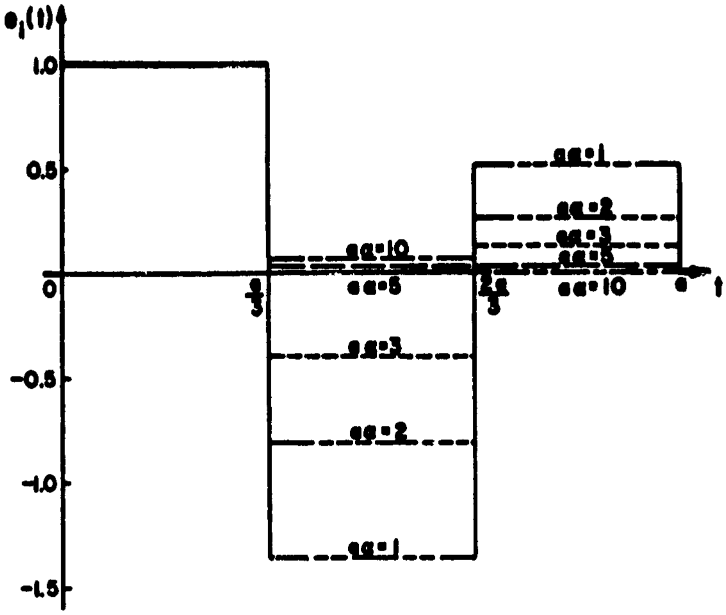
$$e_o(t) = 2\alpha^2(1-\cos\frac{2\pi t}{a})^2\ , \qquad 0 \le t \le a. \tag{5-17}$$

The waveforms are again plotted for several values of $a\alpha$ in Fig. 5.8.

5.3.3 Pulse Transmission Efficiency

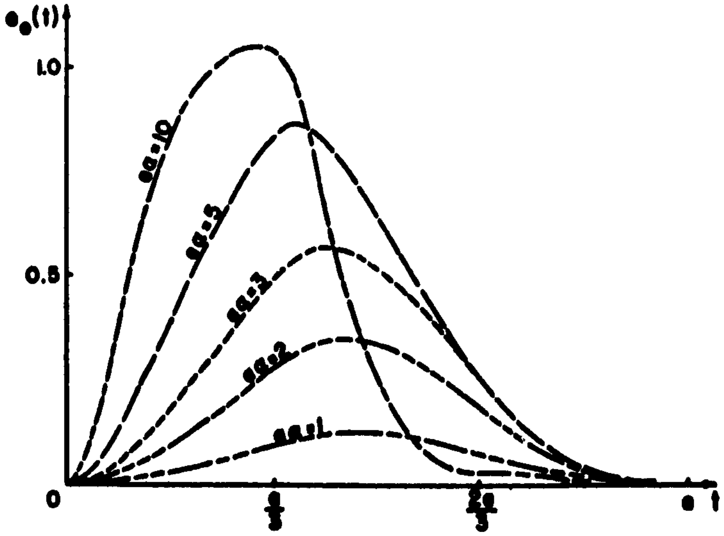
Since for any given network a wide variety of input pulses result in output pulses of the same duration, which of these input pulses are to be preferred over other such input pulses? The answer to this question in general depends on additional specifications regarding the communication system, such as are listed in section 5.2.2. In the specific case under consideration as outlined in section 5.2.3, however, it is desirable to maximize the received energy, which will minimize the error rate in the case of a matched-filter receiver.

A convenient concept for this purpose is the "pulse transmission efficiency," η_p, defined as the ratio of the "energy contents" of the



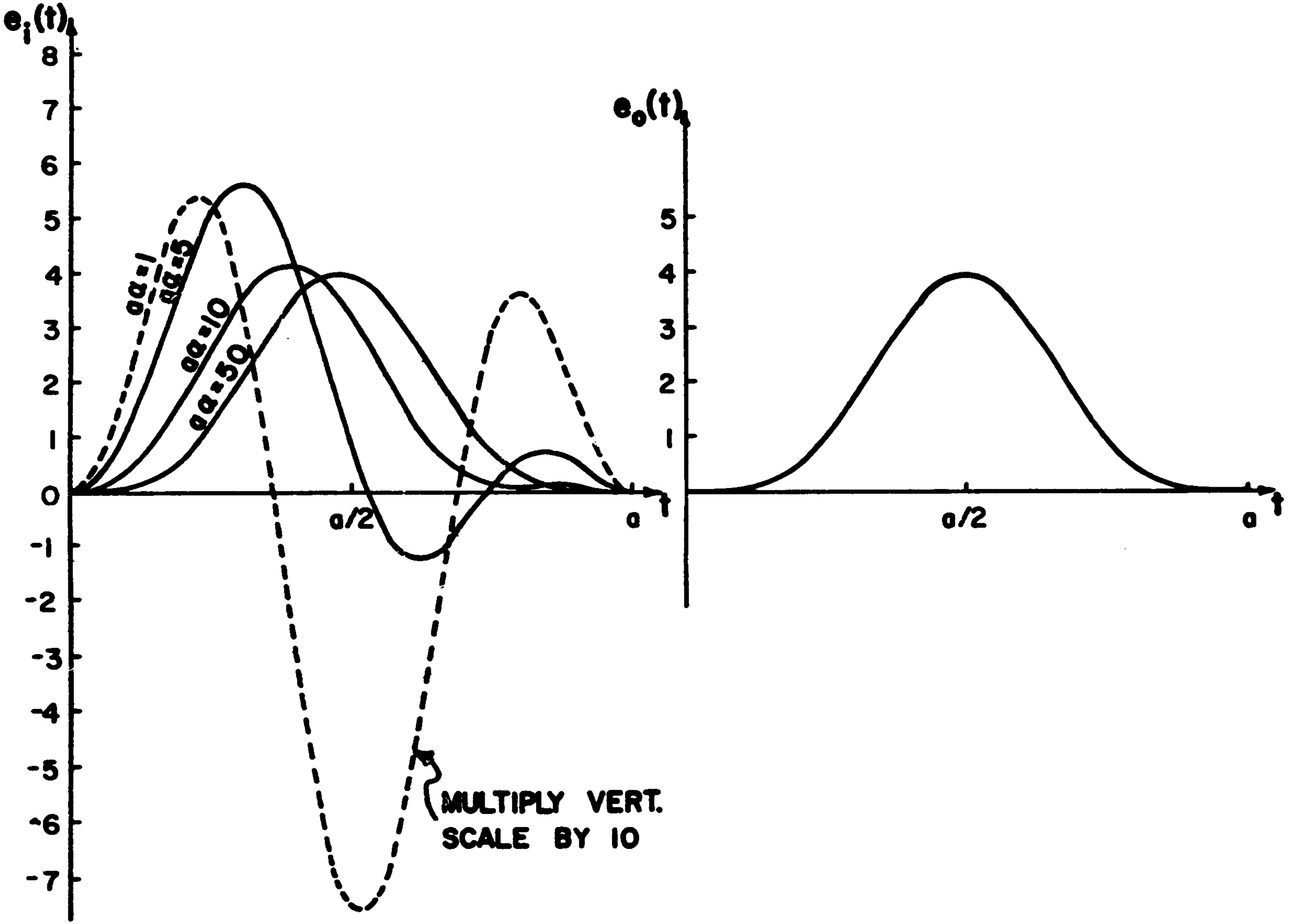
INPUT-OUTPUT PULSE PAIRS OBTAINED IN SECTION 5.3.2.2 (a)

FIGURE 5.7



INPUT-OUTPUT PULSE PAIRS OBTAINED IN SECTION 5.3.2.2 (a)

FIGURE 5.7



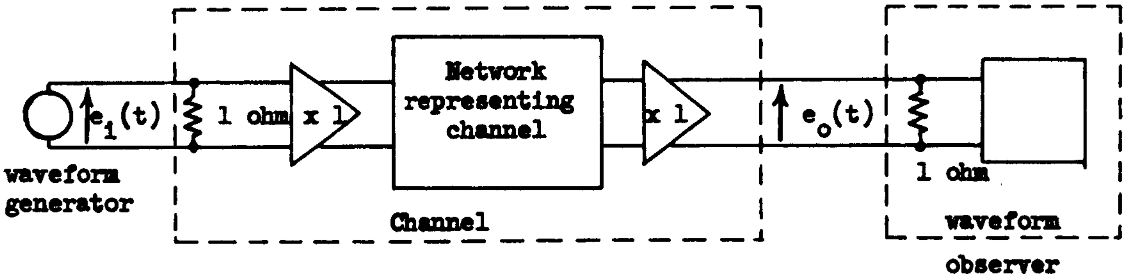
INPUT-OUTPUT PULSE PAIRS OBTAINED IN SECTION 5.3.2.2 (b)

FIGURE 5.8

output and input pulse waveforms, each on a "one-ohm basis":

$$\eta_p = \frac{\int_o^a e_o^2\,(t)\,dt}{\int_o^a e_i^2\,(t)\,dt} \qquad (5\text{-}18)$$

The implication is that the network representing the channel does in fact not present a frequency-dependant input impedance to the waveform generator, (Fig. 5.1) and the waveform observer does not load the channel output. The latter condition can always be maintained by incorporating in the network representing the channel any loading at the channel output. The former condition, however, applies only if the waveform generator is suitably de-coupled from the channel, as would be the case in a radio transmission. Fig. 5.1 might, therefore, be specialized to the following normalized form where the amplifiers have unit gain, infinite input impedance, zero output impedance:



REFINEMENT OF FIGURE 5.1

FIGURE 5.9

Using this representation, $\int_o^a e_i^2\,(t)$ is the energy supplied by the

waveform generator, and $\int_0^a e_o^2(t)$ is the energy delivered to a waveform observer which has 1 ohm input impedance.

5.3.3.1 η_p for the Waveforms Considered in Section 5.3.2.1

Three types of input-output pulse pairs were considered for the RC low-pass network, under a), b), and c) in section 5.3.2.1. The pulse transmission efficiencies for these three types are plotted in Fig. 5.10 as functions of the pulse duration (a) expressed as a multiple of the time constant ($\frac{1}{\alpha}$). It may be noted from these curves that the rectangular shaped input pulse (type "a") results in the greatest energy transfer through the channel for all but very long pulse durations (longer than five time constants).

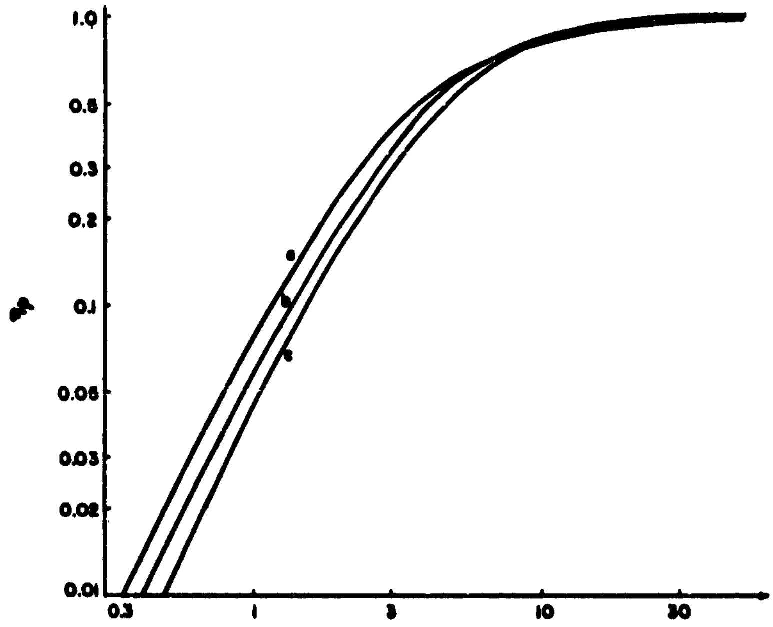
5.3.3.2 η_p for the Waveforms Considered in Section 5.3.2.2

The values of η_p for the two types of waveforms considered under a) and b) of section 5.3.2.2, for the RLC low-pass network, may be plotted as function of the pulse duration similar to the above, and the graph in Fig. 5.11 results. It can be seen that for this network, the rectangular shaped input pulse also results in the greater energy transfer through the network.

5.4 Optimum Waveshapes for Complete Elimination of Intersymbol Interference

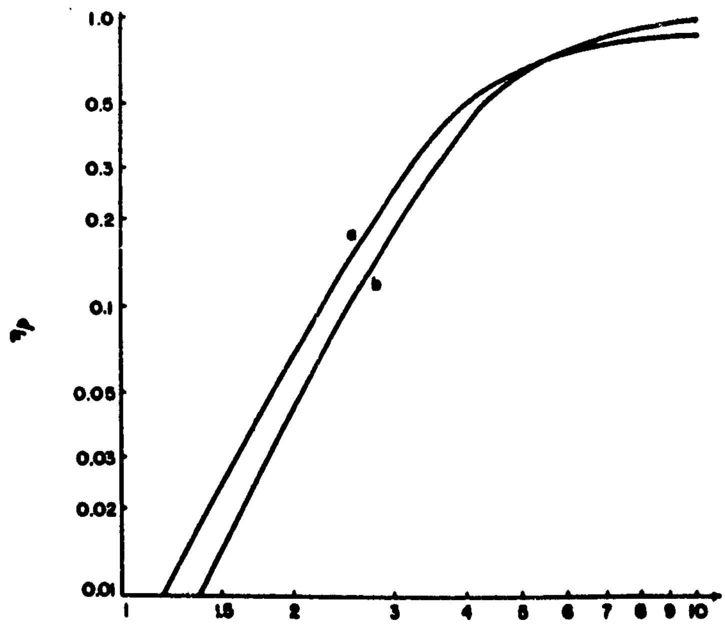
In section 5.3, it was seen that a number of pulse shapes may be applied to a given network so that the output is a pulse of the same duration, but these input pulse shapes in general differ in their ability to transmit energy through the network.

Of the many pulse shapes of specified duration which, when applied



η_p FOR THE WAVEFORMS OBTAINED IN SECTION 5.3.2.1

FIGURE 5.10



η_p FOR THE WAVEFORMS OBTAINED IN SECTION 5.3.2.2

FIGURE 5.11

to a specified network, produce a pulse output, can one be found which maximizes η_p ?

5.4.1 Maximization of η_p

If "Method II" of section 5.3.1 is used to describe the input-output pulse pairs of specified duration associated with a given network, then η_p can be seen to depend only on the choice of the relatively unrestricted auxiliary function, $e_1(t)$. The only requirements on $e_1(t)$ are that it be a pulse which has the specified duration, and it must be differentiable n times, where n is the order of the denominator of the network transfer function.

The Calculus of Variations can, therefore, be applied to find that pulse shape for $e_1(t)$ which maximizes η_p, but with the limitation that in general only 2n-times differentiable functions are admitted as possible solutions.

The expression for η_p is

$$\eta_p = \frac{\int_o^a e_o^2(t)\,dt}{\int_o^a e_1^2(t)\,dt} = \frac{\int_o^a \left[\sum_{i=o}^{m} k_i e_1^{(i)}(t)\right]^2 dt}{\int_o^a \left[\sum_{i=o}^{n} h_i e_1^{(i)}(t)\right]^2 dt} = \frac{N_\eta}{D_\eta} \tag{5-19}$$

Then the first variation of η_p, $\delta \frac{N_\eta}{D_\eta} = \frac{D_\eta \delta N_\eta - N_\eta \delta D_\eta}{D_\eta^2}$ must vanish for all variations δe_1 vanishing at $t = 0$ and $t = a$. Let λ = maximum value of η_p ; then for all such δe_1 , the following condition must hold for the optimizing function $e_1(t)$:

$$\delta N_\eta - \lambda \delta D_\eta = \delta \int_0^a \left[\left(\sum_{i=0}^{m} k_i e_1{}^{(i)} \right)^2 - \lambda \left(\sum_{i=0}^{n} h_i e_1{}^{(i)} \right)^2 \right] dt = 0 \; . \tag{5-20}$$

Then the optimizing function $e_1(t)$ must satisfy Euler's equation of order 2n, in the interval $0 \leq t \leq a$:

$$\begin{aligned}(k_o^2 - \lambda h_o^2)\, e_1(t) + [2(k_o k_2 - \lambda h_o h_2) - (k_1^2 - \lambda h_1^2)]\, e_1''(t) + \\ + \cdots\cdots\cdots\cdots\cdots + \\ +(-1)^{n-1}[2(k_{n-2}k_n - \lambda k_{n-2}h_n) - (k_{n-1}^2 - \lambda h_{n-1}^2)]\, e_1{}^{(2n-2)}(t) + \\ +(-1)^n(k_n^2 - \lambda h_n^2)\, e_1{}^{(2n)}(t) = 0, \; k_i = 0 \text{ for } i > m;\end{aligned} \tag{5-21}$$

with the boundary conditions

$$e_1(0) = e_1(a) = e_1'(0) = e_1'(a) = \ldots = e_1{}^{(n-1)}(0) = e_1{}^{(n-1)}(a) = 0. \tag{5-22}$$

Because boundary conditions are specified at both end points, equation (5-21) is readily solved only for simple cases.

5.4.1.1 RC Low-pass Network

For the network of section 5.3.2.1, Euler's equation becomes

$$e_1'' + \alpha^2(\frac{1}{\lambda} - 1)e_1 = 0. \tag{5-23}$$

The solutions of this equation, satisfying $e_1(0) = e_1(a) = 0$, are of the form $e_1(t) = c \sin\alpha(\frac{1-\lambda}{\lambda})^{1/2} t$, where $\frac{1-\lambda}{\lambda} = (\frac{n\pi}{a\alpha})^2$, n = 1, 2, ...

The value of n which results in the largest λ is clearly n = 1, so that the optimum pulse transmission efficiency in the RC low-pass case is given by

$$\hat{\eta}_p = \frac{1}{1+\left(\frac{\pi}{a\alpha}\right)^2} \tag{5-24}$$

and the optimum waveforms are:

$$e_i(t) = e_1(t) + e_1'(t) = c\left[\alpha \sin\frac{\pi}{a}t + \frac{\pi}{a}\cos\frac{\pi}{a}t\right]$$

$$= \frac{c\alpha}{\cos\arctan\frac{\pi}{a\alpha}}\sin\left(\frac{\pi t}{a} + \arctan\frac{\pi}{a\alpha}\right); \tag{5-25}$$

$$e_o(t) = \alpha e_1(t) = c\alpha \sin\frac{\pi}{a}t. \tag{5-26}$$

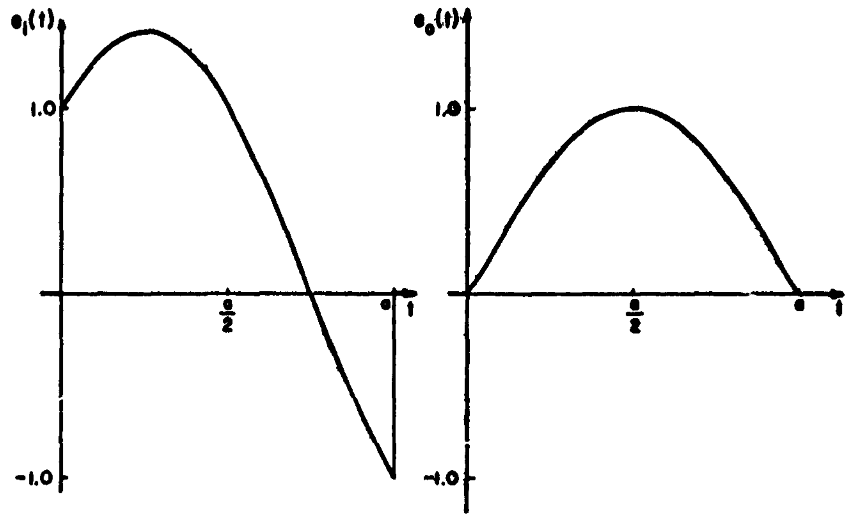
The optimum waveforms are shown in Fig. 5.12 for the case $a\alpha = \pi$, and $\hat{\eta}_p$ is plotted in Fig. 5.13, with the results obtained in sections 5.3.3.1 shown as dotted lines for comparison. For small $a\alpha$, this optimum signal can be seen to result in about a 1 db improvement over the best signal of section 5.3.2.1.

In this first-order case, the variational solution represents an optimization over all those functions $e_1(t)$ whose first derivative exists and is continuous, over the pulse duration. It therefore takes into account all permissible functions $e_1(t)$ except those which contain abrupt changes of slope for $0 \leq t \leq a$. That no function of the latter type can be the optimum $e_1(t)$ can be surmised from the fact that it could be approximated arbitrarily closely by a function with continuous derivative, while the above solution has no suggestion of corners in the interval $0 \leq t \leq a$.

5.4.1.2 RLC Low-pass Network

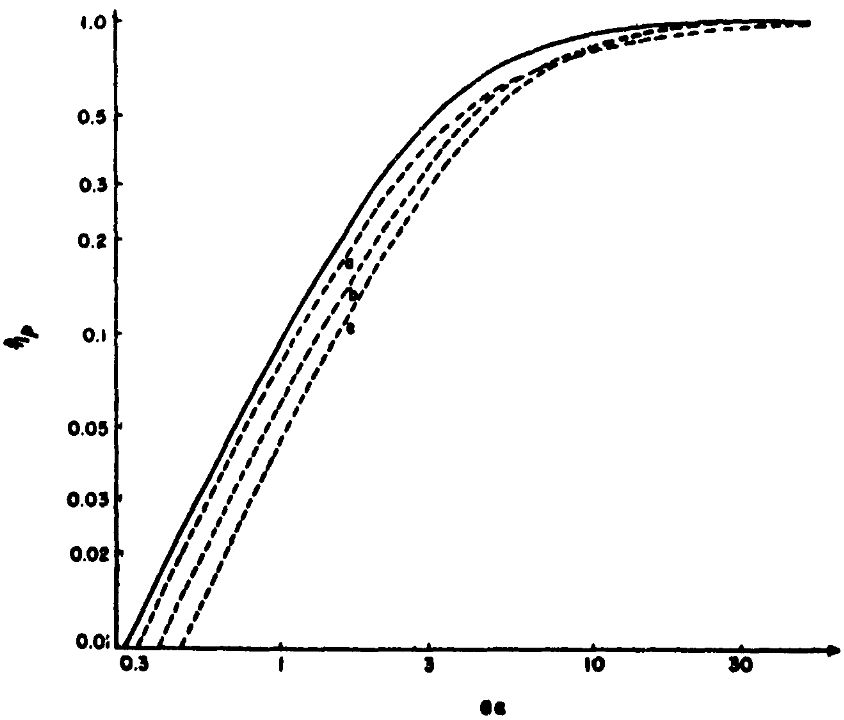
If the variational method is applied to the network considered in section 5.3.2.2, the Euler differential equation becomes:

$$e_1''''(t) + 4\alpha^4(1-\lambda)\, e_1(t) = 0 . \tag{5-27}$$



OPTIMUM WAVEFORM FOR RC LOW-PASS CHANNEL

FIGURE 5.12



$\hat{\eta}_p$ FOR RC LOW-PASS CHANNEL

FIGURE 5.13

The following results are obtained, which are plotted in Figs. 5.14 and 5.15:

$$\hat{\eta}_p \doteq \frac{1}{1 + \left(\frac{3}{2\sqrt{2}} \frac{\pi}{a\alpha}\right)^4} \tag{5-28}$$

$$e_1(t) \doteq c \left\{ [2\alpha^2 - (\frac{4.73}{a})^2] \cos \frac{4.73}{a}(t - \frac{a}{2}) - 9.46 \frac{\alpha}{a} \sin \frac{4.73}{a}(t - \frac{a}{2}) + \right.$$
$$\left. .133[2\alpha^2 - (\frac{4.73}{a})^2] \cosh \frac{4.73}{a}(t - \frac{a}{2}) + 1.26 \frac{\alpha}{a} \sinh \frac{4.73}{a}(t - \frac{a}{2}) \right\} \tag{5-29}$$

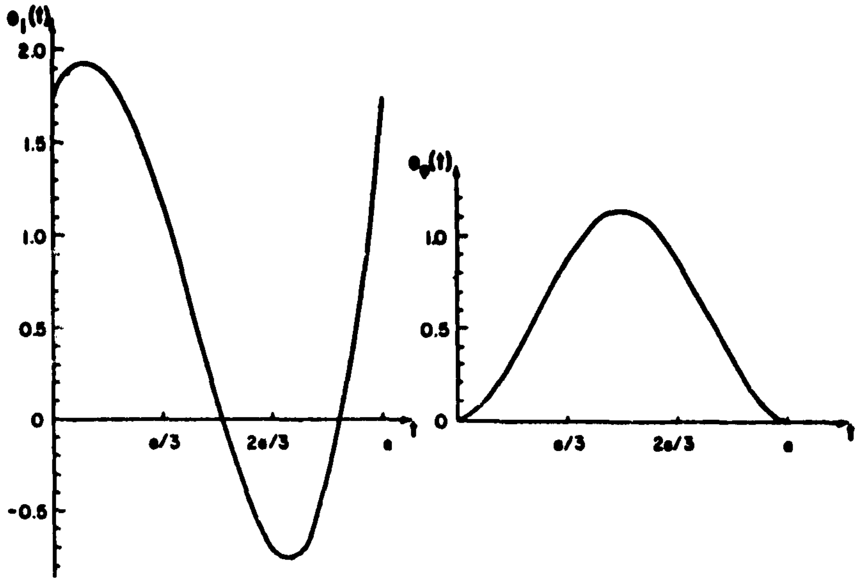
$$e_o(t) \doteq c(2\alpha^2) \left[\cos \frac{4.73}{a}(t - \frac{a}{2}) + .133 \cosh \frac{4.73}{a}(t - \frac{a}{2})\right] \tag{5-30}$$

Fig. 5.15 also shows the results obtained in section 5.3.3.2 as dotted lines for comparison. It can be seen that for small $a\alpha$, the optimum signal (Eq. 5-29) results in an improvement of about 2.5 db over the 3-step signal of section 5.3.2.2.

The above solution represents an optimization over the restricted class of pulse waveforms $e_1(t)$ which are four times differentiable in the range $0 < t < a$.

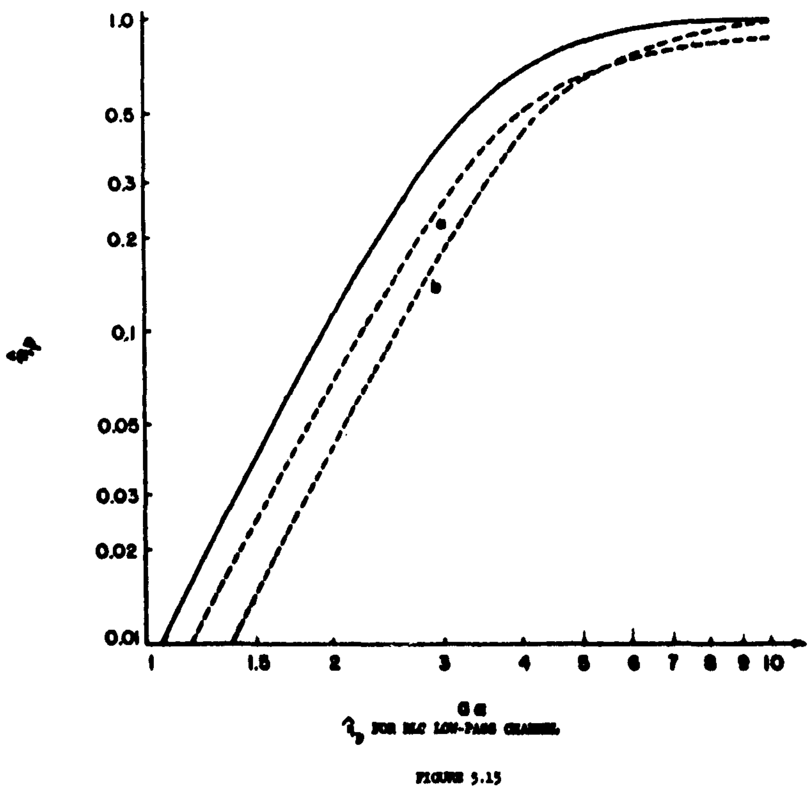
5.4.2 Bandpass Channels

Although only low-pass networks have been considered up until now, the above results may be generalized to bandpass equivalents if the "high-Q assumption" is valid, that is, if the response of the bandpass network is effectively zero at zero frequency. In that case, the optimum pulse waveforms are modulated carriers, with the carrier frequency equal to the center-frequency of the network and the envelope equal to the pulse shape as obtained for the equivalent low-pass case.



OPTIMUM WAVEFORM FOR RLC LOW-PASS CHANNEL

FIGURE 5.14



$\hat{\eta}_p$ FOR RLC LOW-PASS CHANNEL

FIGURE 5.15

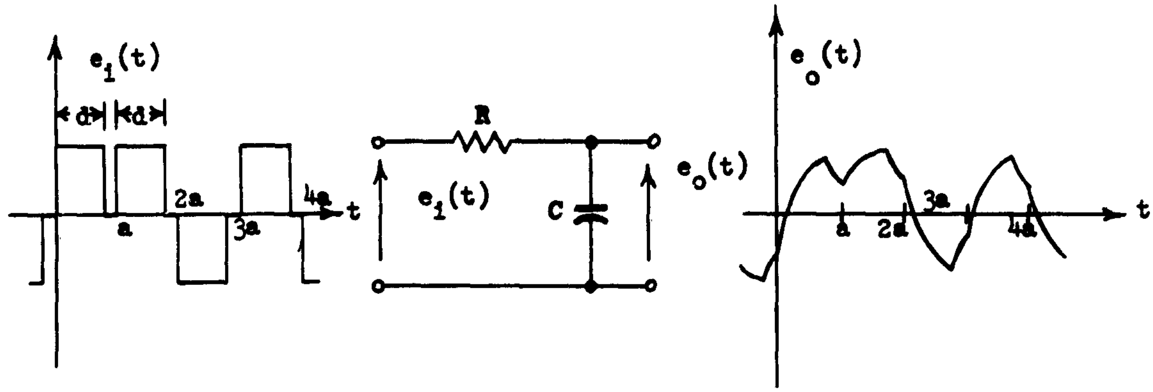
5.5 Comparison with Simple Rectangular Pulse Transmission

The complete cancellation of intersymbol interference has of course been achieved at the expense of a reduction in received energy for a fixed transmitted energy (per pulse). For instance, it can easily be verified, that the gated sine wave signal in section 5.4.1.1 results in a smaller received energy than would a rectangular pulse of equal duration applied to the same channel, given a fixed transmitted energy. But full use of the energy of the rectangular pulse can only be realized if a single pulse is to be transmitted, so that the receiver may observe the exponentially decaying transient over a suitable length of time (which depends on the channel time constant)--i.e., no chance for intersymbol interference.

Thus it is clear that a meaningful comparison must include a consideration of intersymbol interference and transmission rate. For this purpose, a "conventional" transmission consisting of rectangular pulses is appled to the RC low-pass channel, and the performance of this system is compared with the one in section 5.4.1.1.

5.5.1 Simple Rectangular Pulse System

Let the transmitted signal element of duration $\underline{a}$ consist of a rectangular pulse of duration $\underline{d}$, where $d \leq a$. This signal element and its opposite polarity counterpart comprise the binary signal alphabet. The duration $\underline{d}$ is fixed but not initially specified, in order to permit some control over the intersymbol interference by selection of a suitable value for $\underline{d}$. Channel input and output waveforms for a typical transmission of this type are shown in Fig. 5.16:



SIMPLE RECTANGULAR PULSE SYSTEM

FIGURE 5.16

It can be seen that it may be desirable to make d smaller than a, in order to reduce the intersymbol interference. An expression for this interference will now be obtained.

First it is necessary to give a quantitative definition for the intersymbol interference. As previously stated, the waveform observer is assumed to be a matched filter. Its output after every received signal element, and in the absence of interference of any kind, is one of two possible voltage levels of equal magnitude and opposite polarity. By intersymbol interference will be understood the fractional contribution to this voltage level, due to signal energy transmitted prior to the particular signal element intended to be indicated by this voltage.

The intersymbol interference experienced by any received element may thus depend on the polarities of several preceding signal elements. In the computations which follow, the maximum intersymbol interference, denoted by I_m, will always be considered; i.e., the interference which --

for the system being considered -- arises from a string of equal polarity pulses.

E_{01}, the energy received in the interval $0 < t < a$, due to a signal element transmitted during $0<t<a$, is: (assuming pulse amplitude = 1 at channel input)

$$E_{01} = \int_0^d (1-\epsilon^{-\alpha t})^2 \, dt + \int_d^a [(1-\epsilon^{-\alpha d}) \, \epsilon^{-\alpha(t-d)}]^2 \, dt$$

$$= d - \frac{1}{\alpha} + \frac{\epsilon^{-\alpha d}}{\alpha} - \frac{1}{2\alpha} (\epsilon^{2\alpha d} - 2\epsilon^{\alpha d} + 1) \, \epsilon^{-2\alpha a}. \tag{5-31}$$

The value of the output voltage at $t = 0$, due to a single transmitted pulse initiated at $t = -a$, is $e_o(0) = (\epsilon^{\alpha d} - 1)\epsilon^{-\alpha a}$. After another element length this voltage decays to $e_o(a) = (\epsilon^{\alpha d} - 1) \, \epsilon^{-2\alpha a}$; etc. The maximum possible interfering waveform, in the interval $0< t< a$, due to a string of equal-polarity input pulses preceding $t = 0$ is therefore

$$e_x(t) = (\epsilon^{\alpha d} - 1) \left(\sum_{n=1}^{\infty} \epsilon^{-\alpha n a} \right) \epsilon^{-\alpha t}$$

$$= \frac{(\epsilon^{\alpha d} - 1)\epsilon^{-\alpha a}}{1-\epsilon^{-\alpha a}} \, \epsilon^{-\alpha t} , \qquad 0<t<a. \tag{5-32}$$

Contribution by this interference to the output of the matched filter is

$$E_{ox} = \int_0^a e_o(t) \, e_x(t) \, dt$$

$$= \frac{(\epsilon^{\alpha d} - 1) \, ^{-\alpha a}}{1-\epsilon^{-\alpha a}} \left[\int_0^d (1-\epsilon^{-\alpha t})\epsilon^{-\alpha t} \, dt + \int_d^a (1-\epsilon^{-\alpha d})\epsilon^{-\alpha(t-d)}\epsilon^{-\alpha t} \, dt \right]$$

$$= \frac{e_o^2(0)}{2\alpha} \; \frac{\epsilon^{\alpha(a-d)} - \epsilon^{-\alpha a}}{1-\epsilon^{-\alpha a}} . \tag{5-33}$$

The intersymbol interference is therefore

$$I_m = \frac{E_{ox}}{E_{o1}} = \frac{e_o^2(0)\ (\epsilon^{\alpha(a-d)}-\epsilon^{-\alpha a})}{[2\alpha d+2\epsilon^{-\alpha d}-2-e_o^2(0)]\quad(1-\epsilon^{-\alpha a})}\ . \tag{5-34}$$

As in earlier sections, it is again convenient to normalize with respect to α and thus to make $a\alpha$ one variable in the above equation, while $\underline{d}$ can be written as a fraction of $\underline{a}$.

The solid curves in Fig. 5.17 are contours of constant I_m plotted in the $a\alpha$, $\frac{d}{a}$ plane. Some incidental facts about the rectangular pulse system may be noted. It can be seen that as $a\alpha$ decreases, I_m increases rapidly. For small values of $a\alpha$, changing $\underline{d}$ has little effect on the maximum intersymbol interference. However, for any given value of $a\alpha$ (given channel time constant and transmission rate), I_m is always minimized by making $d = 0$. Unfortunately, this means no transmission.

The pulse transmission efficiency of the rectangular pulse system, $\eta_r(d)$, is given by the expression

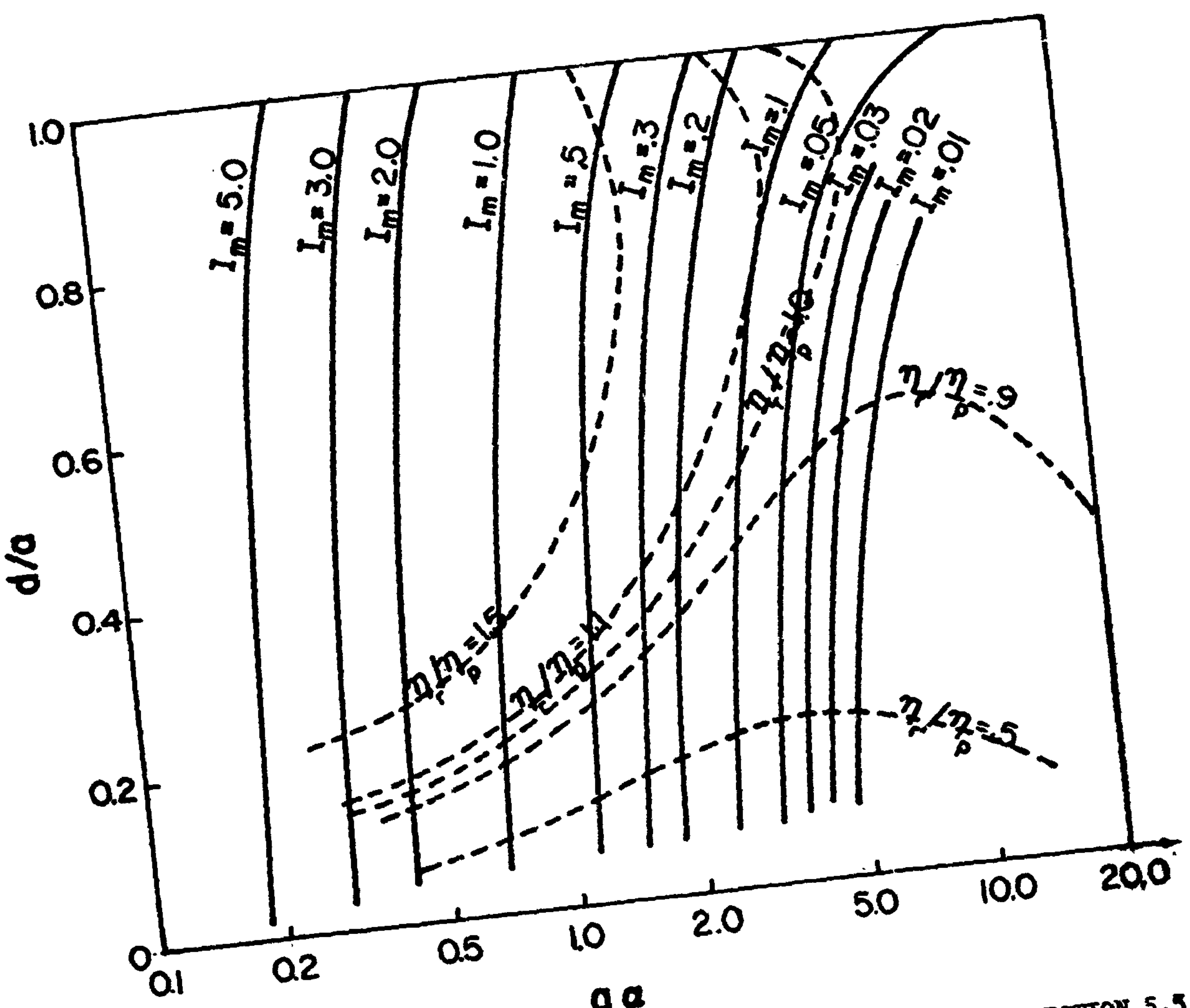
$$\eta_r(d) = \frac{E_{o1}}{E_i} = \frac{E_{o1}(d)}{d}\ . \tag{5-35}$$

This may be compared with η_p for the transmission system in section 5.4.1.1.

<u>5.5.2 Comparison of the Transmission of Section 5.4.1.1 with that of Section 5.5.1</u>

An RC low-pass channel with a certain time constant $\frac{1}{\alpha}$ is assumed given, and it is desired to transmit at a certain rate $\frac{1}{a}$ through this channel; i.e., $a\alpha$ is assumed specified. In addition it is specified that the intersymbol interference may not exceed a certain value.

Two types of transmissions are considered for use in this situation, the gated sinusoid transmission of section 5.4.1.1 and the rectangular



CONTOURS OF CONSTANT I_m and $\frac{\eta_r}{\eta_p}$ FOR THE PERFORMANCE COMPARISON IN SECTION 5.5

FIGURE 5.17

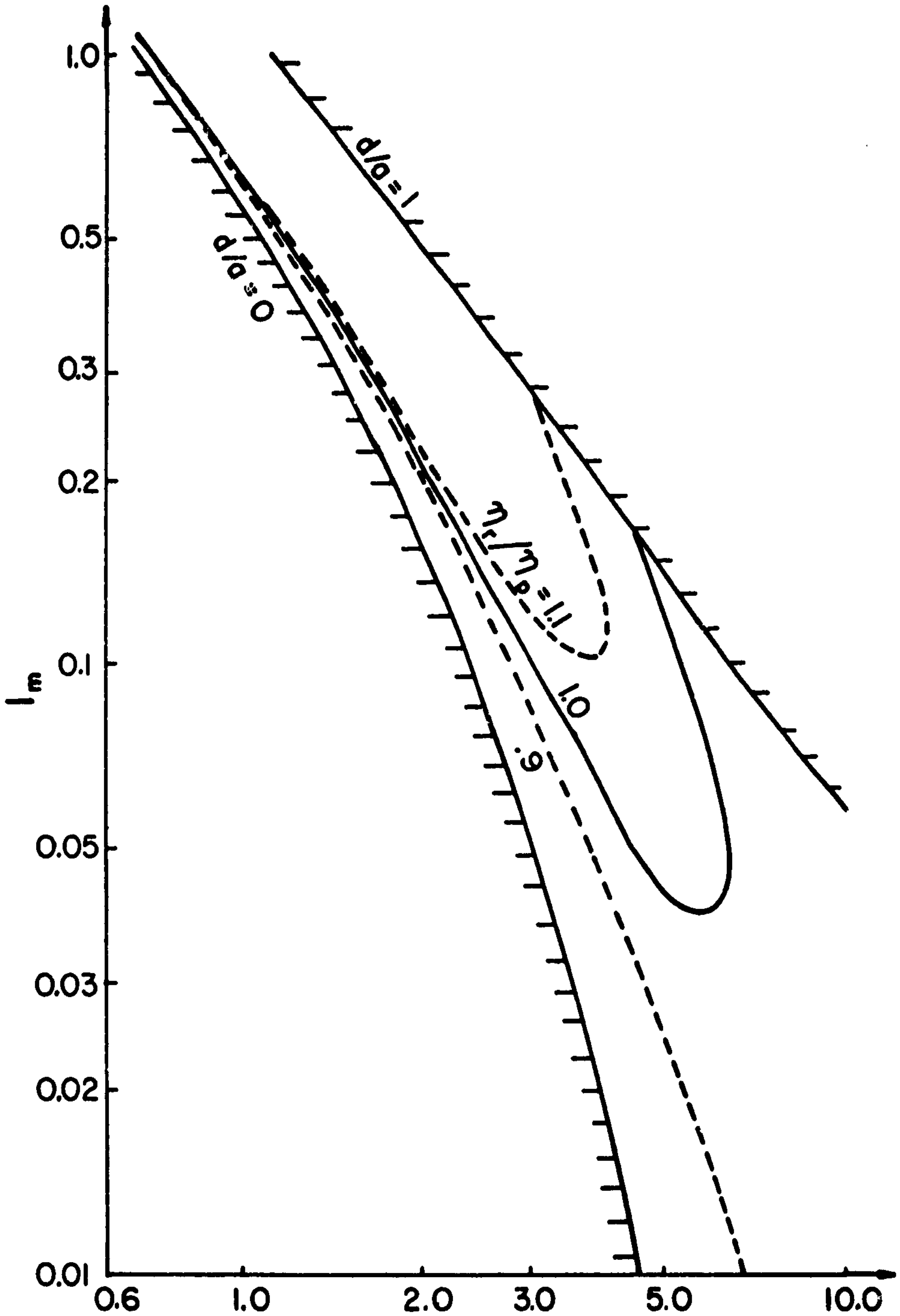
pulse transmission of the previous section, the latter with arbitrary value for d, $d \leq a$. For the specified conditions, how do the pulse transmission efficiencies for the two types of signals compare?

It is merely necessary to consider the ratio $\frac{\eta_r(d)}{\eta_p}$. If this ratio is greater than 1, the rectangular pulse transmission is more efficient; if it is less than 1, the gated sinusoid transmission is more efficient.

Contours of constant values for this ratio are shown as dashed lines in Fig. 5.17. To the left of the contour $\frac{\eta_r}{\eta_p} = 1$, the rectangular pulse transmission is more efficient. Note that this is possible only if about 4% maximum intersymbol interference, or more, is permitted. Thus, if the allowable maximum intersymbol interference is greater than about 4%, and also is such that it can be satisfied by the rectangular pulse transmission for a specified value of $a\alpha$, then d can be adjusted to make the rectangular pulse transmission more efficient. For instance, if $a\alpha = 1.8$ and $I_m = 40\%$ maximum are specified, then transmission of rectangles of duration $\frac{3}{4}$ a is 50% more efficient than the gated sinusoid transmission.

What happens as $\frac{1}{a\alpha}$ -- the product of transmission rate and time constant -- is to be increased, while the maximum tolerated I_m is held constant, can be seen by sliding along the appropriate I_m contour in Fig. 5.17, or by referring to Fig. 5.18, where I_m is read along the vertical axis. Let the specified maximum intersymbol interference be 10%. The performance of the rectangular pulse system can be seen to be as follows:

$a\alpha > 6.5$: $I_m < 0.1$ always, for $d \leq a$ (greatest efficiency is achieved with d slightly less than a; $\eta_r < \eta_p$ slightly).



CONTOURS OF CONSTANT η_r/η_p FOR THE PERFORMANCE COMPARISON

IN SECTION 5.5

FIGURE 5.18

$6.2 < a\alpha < 6.5$: $I_m \leq 0.1$ by suitable selection of d; η_r slightly less than η_p

$3.0 < a\alpha < 6.5$: $I_m \leq 0.1$ and $\eta_r \geq \eta_p$ by suitable selection of d

$2.4 < a\alpha < 3.0$: $I_m \leq 0.1$ but $\eta_r < \eta_p$ for all allowable d

$a\alpha < 2.4$: I_m exceeds 0.1

5.5.3 Summary of Comparison

In summary the following conclusions may be drawn from the above comparison. Consider a fixed channel time constant:

1. If the time allotted to one signal element is sufficiently long (compared to the channel time constant), the gated sinusoid signal is very slightly superior to the rectangular pulse signal.

2. There is a range of element durations in which the rectangular pulse transfers more energy through the channel than does the gated sinusoid, and yet does not produce excessive intersymbol interference. For instance, if no more than 10% maximum intersymbol interference is tolerated, this range is about 2:1, corresponding to $3.0 < a\alpha < 6.2$.

3. For short durations (high transmission rate) the gated sinusoid transmission becomes much less efficient than the rectangular pulse transmission, but the latter results in very large intersymbol interference. In other words, as the transmission rate is increased, the intersymbol interference produced by the rectangular pulse system increases, and it takes an increasing fraction of the transmitted energy to achieve elimination of the intersymbol interference.

5.6 Sensitivity of the Optimum Performance to Changes in Channel Parameters

After the optimum input pulse - one which maximizes the energy transfer through the given channel - has been found, it is of interest to determine

the effect of slight changes in the channel characteristics. In such a case, the output is generally no longer a pulse, and consideration must be given to the energy received during the intended pulse duration, as well as the energy received thereafter due to the remaining transient, the sum of the two being the total received energy for $t > 0$.

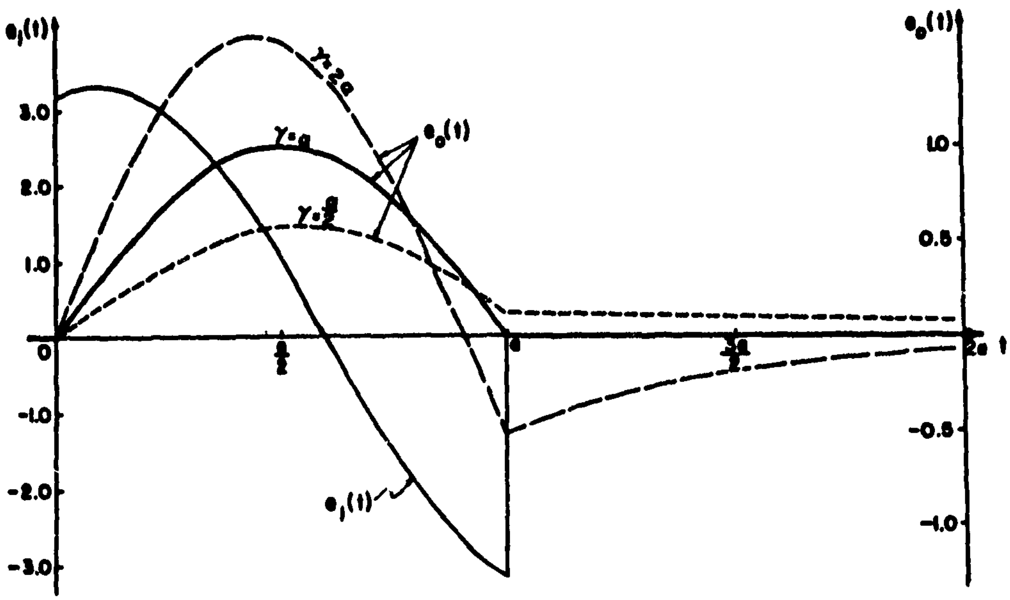
A change in the channel parameters (or their inaccurate determination) thus affects system performance not only by a change in the received energy, but also by the introduction of intersymbol interference which had been thought eliminated. Besides, the received waveform also changes, so that the "waveform observer" would have to be matched to a new waveform in order to utilize fully the received energy. This latter problem is not considered in this section, but the received energies have been computed for a particular case.

5.6.1 The Pulse of Section 5.4.1.1 Transmitted Through an Arbitrary RC Low-pass Network

In section 5.4.1.1 the input pulse waveform of specified duration was found which effects the most efficient energy transfer through an RC low-pass network of time constant $\frac{1}{\alpha}$ and results in a pulse at the output. If this input waveform is applied to an RC low-pass network with time constant $\frac{1}{\gamma}$, then the following observations can be made:

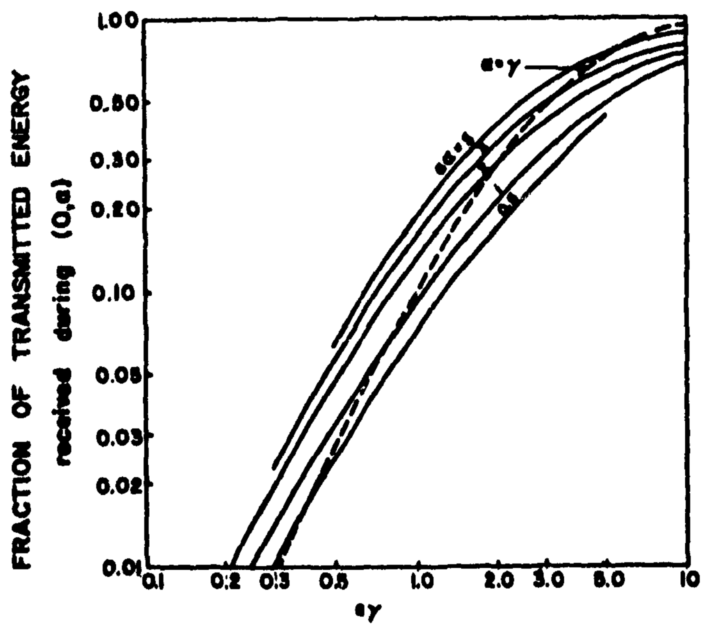
a) The output waveform is a pulse only if $\alpha = \gamma$. (Fig. 5.19)

b) For a given transmitted energy the energy received during the interval $0 \leq t \leq a$ increases with γ, but for $\gamma > \alpha$ it is less than it would be if the transmission were optimized for γ, whereas for $\gamma < \alpha$ it is greater than what it would be if the transmission were optimized for γ. (Fig. 5.20)



PULSE OPTIMIZED FOR CHANNEL TIME CONSTANT $\frac{1}{a}$, AFTER TRANSMISSION THROUGH A CHANNEL WITH TIME CONSTANT $\frac{1}{\gamma}$

FIGURE 5.19



RECEIVED ENERGY DURING (0, a) WHEN THE CHANNEL TIME CONSTANT IS $\frac{1}{\gamma}$ AND THE PULSE IS OPTIMUM FOR A CHANNEL TIME CONSTANT $\frac{1}{a}$.

FIGURE 5.20

c) For $0.5 \leq \frac{\tau}{\alpha} \leq 2.6$, approximately, the energy received <u>after</u> the time interval (o, a) is always less than 10% of the energy received <u>during</u> (o, a). For $0.8 \leq \frac{\tau}{\alpha} \leq 1.3$, approximately, the energy received <u>after</u> the time interval (o, a) is always less than 1% of the energy received <u>during</u> this interval. (Fig. 5.21)

More detailed information may be taken from the accompanying graphs which give the results of the computations performed. It may be concluded that the performance of the system of section 5.4.1.1 is not very sensitive to small changes in channel time constant.

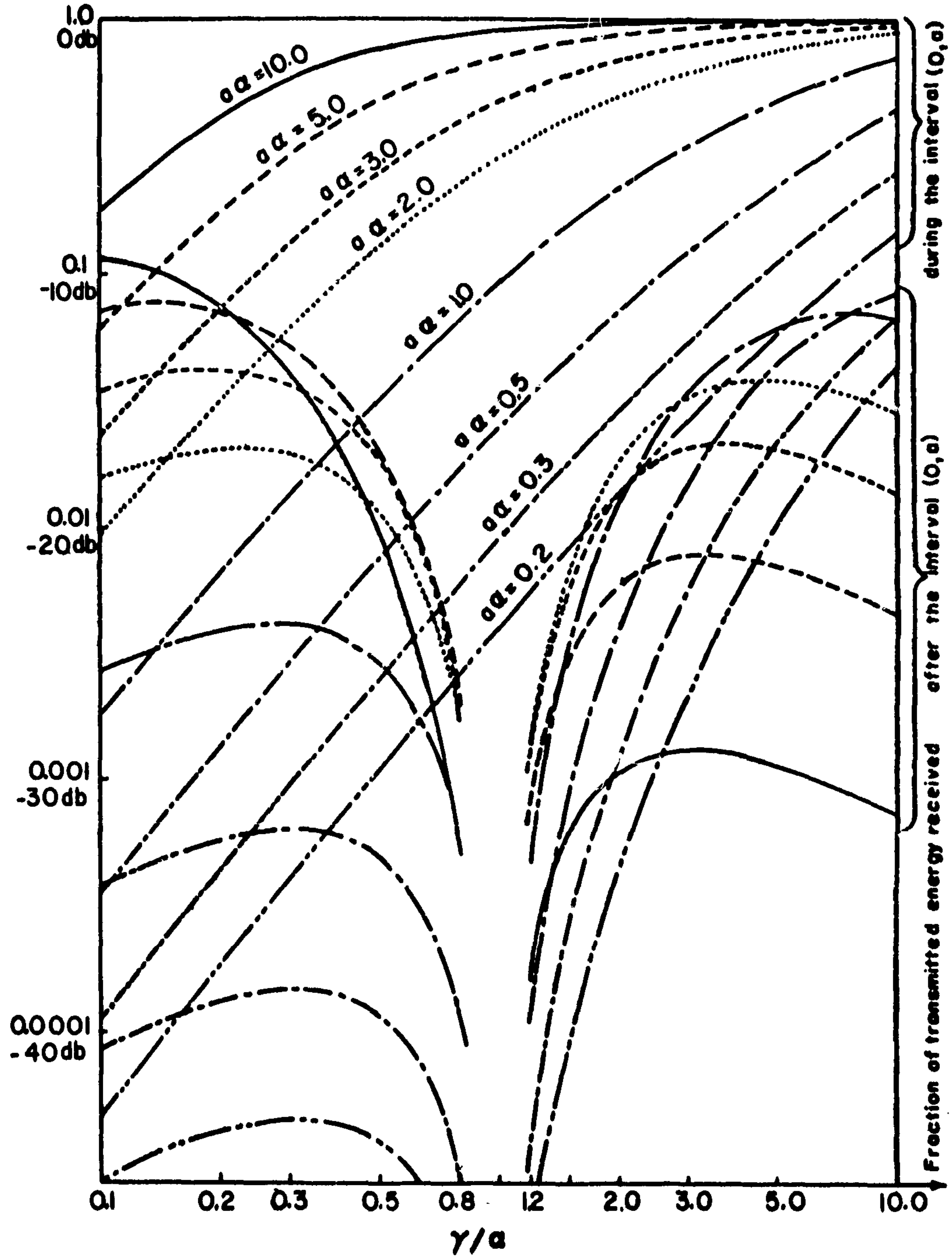
5.7 Transmission of Overlapping Pulses

In this section, a mode of pulse transmission is considered which differs from the one implied in the discussion up till now. It will be shown that a further improvement over the optimum transmission of section 5.4 is possible.

So far, it has been assumed that signal energy which is transmitted <u>during</u> the interval (o, a) but received after time t = a causes intersymbol interference, i.e., the next signaling element is transmitted and received in the interval (a, 2a). Instead, pulses are now transmitted so that their durations partially overlap, the region of overlap being specified. The receiver is assumed to make no observations during the interval of overlap.

The same kind of channel is assumed as has been considered in previous sections.

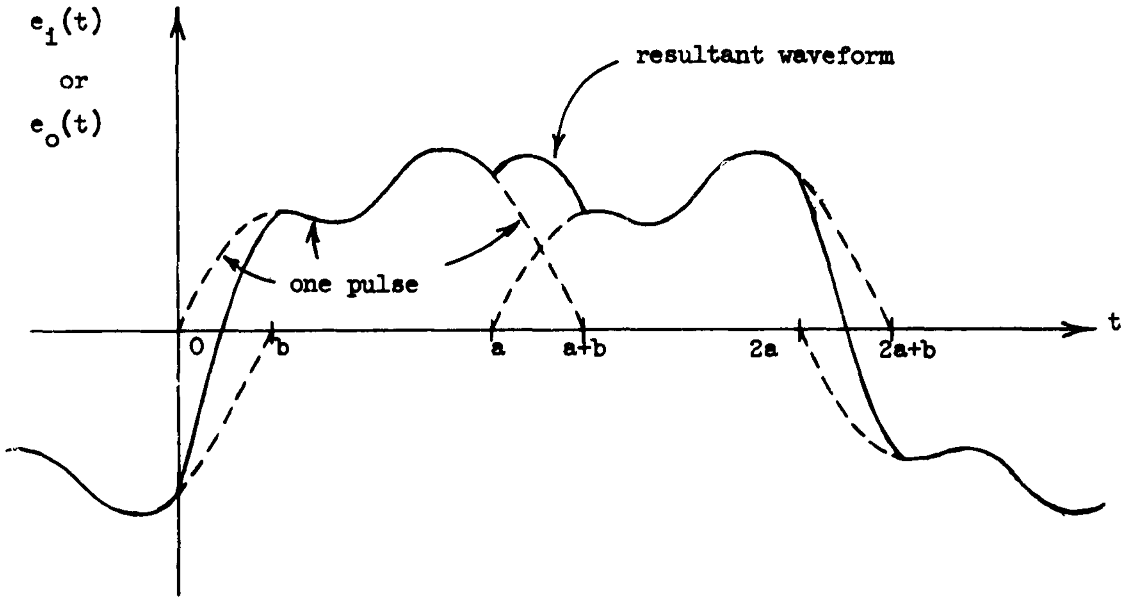
In order to make the results obtained here commensurate with those of the earlier sections, the transmission rate, $\frac{1}{a}$, should remain the same; i.e., new pulses are initiated every <u>a</u> seconds. The pulse duration



COMPARISON OF ENERGY RECEIVED DURING AND AFTER THE INTERVAL (0, a)

FIGURE 5.21

is, therefore, taken to be a + b, where b is the interval of overlap, as indicated in Fig. 5.22.



PULSE TRANSMISSION WITH OVERLAP

FIGURE 5.22

Since the waveform observer is only operating during the interval (b, a), the performance criterion becomes the ratio

$$\eta_p(b) = \frac{\text{received energy during the interval (b, a)}}{\text{average transmitted energy per pulse}} \tag{5-36}$$

Because the waveforms also overlap at the transmitter, the denominator of the above expression requires some additional assumptions. Let it be assumed that successive transmitted pulses are selected independently with equal probability from an antipodal binary waveform alphabet. In that case,

$$\eta_p(b) = \frac{\int_b^a e_o^2(t)\,dt}{\int_b^a e_i^2(t)dt + \frac{1}{2}\int_o^b [e_i(t) + e_i(t-a)]^2 dt + \frac{1}{2}\int_o^b [e_i(t)-e_i(t-a)]^2\,dt}$$

$$= \frac{\int_b^a e_o^2(t)\,dt}{\int_o^{a+b} e_i^2(t)\,dt} \tag{5-37}$$

5.7.1 The Pulse of Section 5.4.1.1 Transmitted with Overlap

As a specific example, the optimum system of section 5.4.1.1 will now be called upon to transmit at some rate $\frac{1}{a}$ with some overlap b, $0 \leq b < a$. Is it possible to achieve an improvement over the performance obtained in section 5.4.1.1?

For this system,

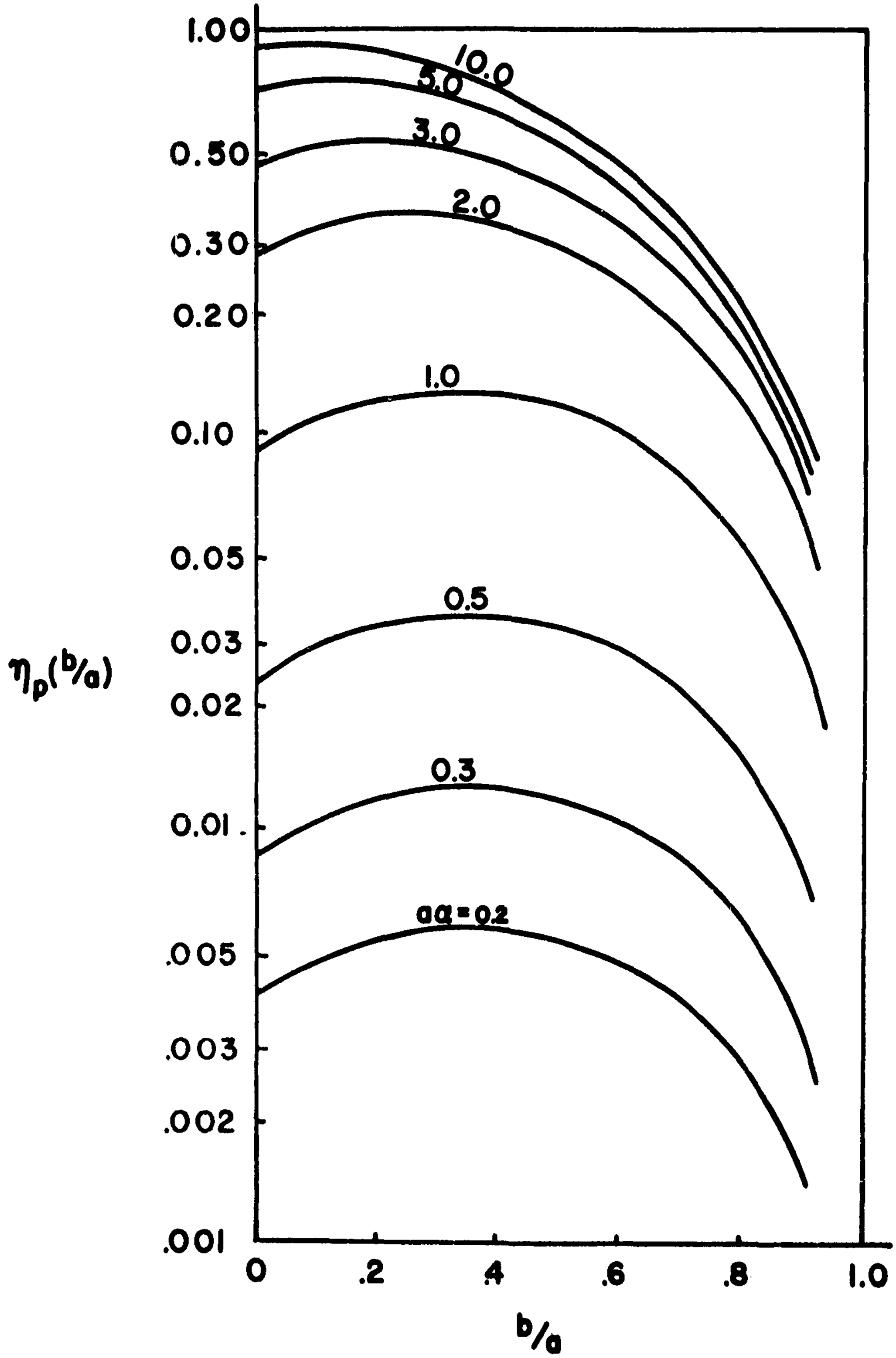
$$\eta_p(b) = \frac{(a^2-b^2) + \frac{1}{\pi}(a+b)^2 \sin \frac{2\pi b}{a+b}}{(a+b)^2 + \frac{\pi^2}{\alpha^2}} \tag{5-38}$$

A plot of this expression, for different values of $a\alpha$, is given in Fig. 5.23 and indicates that a non-zero value for b can improve the energy transfer through the system, in spite of the fact that some of the received energy is deliberately discarded.

This shows that further optimization of the transmitted signal is possible (beyond the optimum obtained in section 5.4) while still avoiding intersymbol interference.

5.8 Conclusions

The investigation reported in this chapter shows that definite



PULSE TRANSMISSION EFFICIENCY USING OVERLAPPING PULSES, AS FUNCTION OF THE FRACTION OF OVERLAP

FIGURE 5.23

improvements can be achieved in the performance of a communication system by giving suitable consideration to the design of signals. An alternate benefit to be derived from an application of signal design would be the easing of coding requirements while maintaining the same system performance.

Optimum pulse signals have been found for non-overlapping transmission which satisfy the requirment of zero intersymbol interference at the receiver. This optimization has been made for arbitrary signaling rates. The signals obtained in this manner for a given channel can be used for transmission at rates that are sufficiently high to prohibit the use of simple rectangular pulses because these cause excessive smearing of the received waveforms.

It has been shown that for a simple channel model the performance obtained with signals that are optimized for this channel does not degrade rapidly with changes in the channel characteristics. This is of particular interest in establishing requirements for channel identification measurements.

Finally it has been shown that further performance improvement is possible by permitting successive transmitted waveforms to overlap somewhat.

Only very specific cases have been examined in some detail in this preliminary study. However, the results obtained give some insight into the properties and behavior of signals in digital communications. They also point out the need for much more work in this area. More theory must be developed to treat the problem of signals design, while the results to be obtained are almost certain to greatly benefit the communications art.

Further investigations should specifically be concerned with the following topics:

1) Continuation of the work presented in this chapter, that is, the optimization of transmission for the system model as described in section 5.2.3.

2) The application of other performance criteria, such as given in section 5.2.2.2, suitably related to practical system requirements.

3) Consideration of models for more general types of channels, as listed in section 5.2.2.1, which also includes the problem of specifying appropriate channel models on the basis of specified practical system parameters.

CHAPTER VI

PERFORMANCE OF ERROR CORRECTING CODES

6.1 Introduction

An important method of increasing the reliability of digital data transmission systems is the coding of the information to be transmitted in such a manner as to enable the receiver to detect and possibly correct the more probable error patterns that the channel may introduce. A brief heuristic discussion of the philosophy of coding for error reduction appears later in this chapter.

Many coding/decoding schemes, of varying complexity and capabilities, have been proposed; it is standard to express the capability of a code in terms of the types and magnitudes of the error patterns which that code will detect, or detect and correct. However, such expressions of capability are useful in the analysis of the "goodness" of the code only with reference to other codes of similar complexity; they do not allow comparison of the performance of an uncoded channel to that of a channel utilizing the code.

It is the intention of this chapter, then, to explore the relative advantages (principally, an increase in reliability) of coded versus uncoded systems, and the costs (in the most general sense) of attaining these advantages. Although the form of a general solution valid for all codes of the type studied is presented, analytic and numerical results are obtained only for the more easily implemented codes.

6.2 Outline

This chapter is divided into several sections. A brief outline of the contents of each section follows.

Section 6.3 presents briefly a discussion of the field of error reduction coding. Much of the mathematics involved in the formulation of error correcting and error detecting codes is omitted; however, sufficient detail is included to enable the reader unfamiliar with the terminology to follow the remainder of the chapter.

Section 6.4 discusses the parameters involved in assessing the quality, from performance standpoint, of coding schemes; a measure of code merit is postulated and discussed in the last part of this section.

Section 6.5 presents and discusses the restrictions introduced upon the systems to be analyzed in detail. There are: a binary system, a symmetric memoryless source and a symmetric memoryless channel disturbed by additive white Gaussian noise.

In section 6.6, a brief resume of the relationships between channel signal-to-noise ratio and the binit rate is presented.

Section 6.7 relates the channel probability of error to the binit probability of error at the decoder output. The general solution is a variation of a form found in the literature, as is the philosophy of the computer simulation method of solution; the analytic solution for the Hamming codes, however, is new.

The numerical results are presented in detail in Section 6.8; the accompanying text explains the exact interpretation of the graphs, and includes examples of their use.

Mathematical derivations in the body of the report are reduced to a minimum; when these are available in other publications, reference is made through the bibliography. The derivation of the Hamming code error rate equations is new, and is presented in detail in Appendix IV. The

results of computer simulation are presented in Appendix V. Tables of coefficients for the Hamming code error rate equations are included as Appendix VI.

6.3 Coding for Error Reduction

6.3.1 Introduction

It is the intention of this section, not to detail with mathematical precision the various methods and philosophies of error reduction coding, but to present heuristically, with a minimum of such mathematics, a general discussion of the field. For detailed or mathematical discussions of coding theory and specific codes, many excellent references are available.

Shannon(33) in his treatment of the theory of communication, proves that information may be transmitted over a noisy channel with arbitrarily low error rate provided the rate at which such information is supplied to the transmitter is lower than the channel capacity, or, in other words, providing that there is room for the insertion of redundancy. A very simple example of such redundancy insertion is a system that transmits every binary digit, or "binit", three times; the observer (i.e., the decoder) at the receiver assumes that the actual transmitted binits all had the same value as that of the largest number of identical received binits. Such a system interprets correctly, then, any error pattern which results in either zero or one error in every block of three binits corresponding to a single transmitter input binit.

However, in this example, three binits are used to convey the information originally contained in one -- obviously a very high sacrifice of channel capacity. The search for better codes may be described as a search for efficient methods of introducing redundancy into the information to be transmitted.

6.3.2 Group Codes

This coding review will deal with group codes only. Group codes have several interesting characteristics; their main distinguishing feature, however, is the general encoding and decoding method. The information binits supplied to the transmitter are accepted in fixed length blocks. To each such block is adjoined a fixed number of check binits, whose values are determined by the information binit values, forming a code word. Similarly, at the receiver, the incoming stream of binits is broken up again into code words (note that synchronization is required -- each received word is a transmitted word, except for binits changed, and thus in error, by the noise in the channel). Each code word is then interpreted, after the correction procedure is completed, as a representation of a particular block of information binits.

Another characteristic of group codes is that the set of all code words forms a vector space, where the individual elements of each vector (code word) are elements from the modulo 2 field (in the modulo 2 field, $0 + 0 = 0$; $0 + 1 = 1$; $1 + 1 = 0$). Thus, the vector addition of any two code words is also a code word.

6.3.3 The Decoding Table

There are many ways of representing a particular group code; perhaps the most straightforward and complete, however, is the decoding table. The decoding table is a rectangular array of all possible received words; the code words appear in the top row, with the all-zero word (always a member of the set of all code words) at the top of the first (left hand) column. The remainder of the words appear exactly once each in the remainder of the array.

The rules for setting up the array are as follows; to form the i^{th} row (assuming rows 1, 2. ..., i-1 are already formed), place any word not yet used in any previous row in the first column. Then, in each of the other columns, place the word resulting from the vector (modulo 2) addition of this first column entry and the code word heading each cloumn.

Consider the possibility of a word appearing more than once in the table. Allow $\oplus$ to represent vector (modulo 2) addition; set ϵ_1 = the word in column 1, row i, with ϵ_j similarly defined, and $i < j$. Set also, ω_1 = any code word, and ω_2 = also any code word. Assume now, that some word appears twice in the table; in particular, let the entry in row i under ω_1 = the entry in row j under ω_2; then $\epsilon_i \oplus \omega_1 = \epsilon_j \oplus \omega_2$.

Notice, now, that $\omega_2 \oplus \omega_2 = 0$, where 0 represents the vector (word) with all zero entries; also, $\omega_2 \oplus 0 = \omega_2$; then, "adding" ω_2 to both sides

$$\epsilon_j = \epsilon_i \oplus \omega_1 \oplus \omega_2$$

but, for a group code, $\omega_1 \oplus \omega_2 = \omega_3$, some code word; then $\epsilon_j = \epsilon_i \oplus \omega_3$ i.e., ϵ_j already appears in a previous row (in particular, in the i^{th} row under ω_3). Such a choice of ϵ_j as the first word of the j^{th} row would violate the rules for forming the table. Thus, the situation of

$$\epsilon_i \oplus \omega_1 = \epsilon_j \oplus \omega_2, \text{ with } i \neq j,$$

cannot occur. Also, if $i = j$, then $\omega_1 = \omega_2$ -- and this defines one and the same position in the table.

The rows of the decoding table are normally given the name "cosets"; the entry in the first column in each row is termed the "coset leader". Note, now, that a received word must be either a code word or the "sum" ($\oplus$) of a code word and a coset leader. Thus, if the decoder is designed to search this table for a given received word and change the received

word to the code word heading the column in which the received word was found, the decoder is, in effect, making the assumption that the error pattern introduced in the transmitted word by the channel is the coset leader for the coset containing the word actually received. In brief, the error patterns corrected by any given code are coset leaders of the corresponding decoding table.

When, in the formation of the decoding table, the additional rule is introduced that the word chosen for a coset leader is a word of least "weight" (weight = number of 1's) among those yet to be used, the table is said to be in standard array.

6.3.4 Perfect, Quasi-Perfect Codes

A perfect t-error correcting group code is a code that corrects all patterns of t or fewer errors in a code word, but no others. A quasi-perfect t-error correcting code is one that corrects all patterns of t or fewer errors and some patterns of t+1 errors, but no others.

Equivalent definitions would be that perfect t-error correcting codes have as coset leaders all patterns of weight t or less, and no others, while quasi-perfect codes have, in addition, some coset leaders of weight t+1.

6.3.5 Hamming Codes[34]

The basic Hamming codes of length $n = 2^m - 1$ binits correct any word received containing, at most, one error; they are perfect codes and, as such, have all coset leaders (other than the first) of weight one. Thus, an n binit word length Hamming code has n + 1 cosets. The number of information binits is $k = n-m$, leaving m check binits.

One particularly interesting way of encoding the message results in a simple decoding scheme without using a decoding table. Consider

the ordered binary numbers from 1 to n, written with m places (i.e., for m = 3: 001, 010, 011, 100, 101, 110, 111). Let the i^{th} number correspond to the i^{th} binit in the n-place code word. Notice that there are m binits whose binary position representation contains exactly one 1; let these be the m check binits.

Now select all those binits whose binary position equivalent contains a 1 in the "first" (right hand) position -- namely, 1, 3, 5, 7, ..., n-2, and n; let this be the first "check sequence". Similarly the second check sequence is to be made up of those whose binary equivalents contains a 1 in the second position, and so on. Now each check sequence contains, as its first binit, one of the check binits, <u>and no other</u>. Form the code word, then, by filling in arbitrarily all except the check binits; sum (modulo 2) the value of the binits in each check sequence, omitting the check binit, and enter this sum as the corresponding check binit. The sum over any complete check sequence is then zero.

Then, in decoding, again sum the binits in each check sequence. Interpret each sum (modulo 2) as the entry in the corresponding position of an m place binary "check" number. If one error (i.e., a 0 changed to a 1, or a 1 to a 0) had occurred during transmission, a little investigation will show that the resulting check number is the binary position equivalent of the binit in error.

The basic SEC (<u>S</u>ingle <u>E</u>rror <u>C</u>orrecting) Hamming code may be modified so as to detect, without correction, all double errors as well as many of higher order. Consider adding another check binit to a Hamming SEC code word; the value of this binit is 1 if the weight of the basic word

is odd, and 0 if the weight is even. Now, for any double error, the check word may be non-zero (thus locating the error) or zero (indicating that it is the overall check binit that is in error).

6.3.6 Bose-Chandhuri Codes [35, 36]

A full treatment of these codes would not be in keeping with the intent of this report. Suffice it to say that Bose and Chandhuri have devised a general method for constructing codes capable of correcting up to and including t errors, t being any positive integer, and that it has been shown[37] that two-error B-C codes are quasi-perfect, while B-C codes with $t \gtrsim 3$ are not.

6.4 Characteristics of Code Performance

The characteristics of performance referred to are not those technical details associated with the coding-decoding processes; these details are characteristics of the code itself and, although they indicate in a general sense the correction capabilities of the code, they cannot be used as measures of merit or performance. What is meant by the code performance characteristics are the overall measures of the advantages gained by the use of the code, the cost of attaining these advantages, and the merit of the code. (A measure of merit is defined below.)

6.4.1 Costs

6.4.1.1 Complexity

Associated with any code are the mathematical manipulations required to code the input information, and to correct, as applicable, and decode the coded messages at the receiver. Generally, the coding schemes may be implemented with relative ease; the decoding/correction methods, however, range from the relatively simple to the extremely complex.

6.4.1.2 Information Rate Reduction

The information contained in a received sequence of independent digits is a function of the *a-priori* transmitter probabilities for the digit values and the probability of an error being introduced during transmission. It may appear that, for fixed *a-priori* probabilities for the information digits, a code designed to reduce the probability of error would result in an increase in the information rate; however, *for a fixed digit transmission rate*, this increase is, for the low initial error probability case of interest, negligible compared to the reduction in the rate caused by the code redundancy. Thus, for the fixed bandwidth (or constant transmitter rate) case, the net change in the information rate is a decrease, and must be considered as a cost.

6.4.1.3 Omissions

This cost arises only when error-detecting codes are used with one-way channels. In such a situation, a message received in error may be assumed to fall into one of three categories; the error pattern is either one which the code is designed to correct, one which the code is designed to detect without correction, or one which is beyond both the correction and detection abilities of the code. In this latter case, the pattern will normally be interpreted incorrectly by the decoder as being a different correction or detection-without-correction error pattern. Thus, insofar as the decoder is concerned, all received patterns are either correctable, or non-correctable. Although the action to be taken by the decoder upon the detection of a non-correctable error pattern is part of the decoding procedure, those actions will have significant effects on code performance. In the analysis to follow, it is assumed that those re-

ceived words containing detectable but non-correctable high order error patterns will be discarded; the resulting probability of an information binit being discarded, or the omission rate, is investigated in detail in this chapter.

6.4.1.4 Delay

The decoding of any group code requires that the complete code word be available; thus, there can be no output from the decoder until the entire word is received. Except in special situations, this delay is too short to be of significance in the evaluation of code performance.

6.4.2 Advantages

6.4.2.1 Reliability

Ignoring the insignificant increase in information rate resulting from a reduced probability of error (as discussed in 6.4.1.2), decreasing the probability of error for the received information binits at the decoder output, and thus increasing the reliability to be placed in the received data, is the only reason coding would be used.

6.4.3 Measure of Merit

In general, under the constraint of fixed energy/binit and fixed binit transmission rate, coding will buy an increase in reliability at the price of a reduction in information rate. Having two parameters of performance for each code makes comparisons of the value of different coding schemes difficult.

Another system eliminating this difficulty may be postulated. Consider the application of coding to a channel for which the average power and the maximum allowable error rate are specified as design requirements. The error rate required then may be used to calculate the required ratio of the energy per binit to the noise spectral density,

E/N_o, for coded as well as uncoded systems. From these ratios and the fixed average power limitation, a maximum rate of information binit transmission, relative to that for the uncoded system, may be obtained. Such a quantity is well suited for use as a criterion of comparison among different coded, as well as uncoded, systems; it yields directly the changes in the rate of transmission of information binits resulting from the use of error correcting codes.

It should be remembered that for the small error probabilities of interest, the information binit transmission rate is very nearly the information transmission rate of the system. Thus, another proposed criterion, the ratio of information rate to bandwidth, is a function of the number of redundant binits per code word only; these values are supplied in tabular form.

6.5 Restrictions Introduced

As is implied in the chapter title and in the preceding discussions, the major restriction imposed is that of a binary system. In addition, the following restrictive assumptions are made.

6.5.1 Symmetric Memoryless Source

It is assumed that the information to be transmitted has already been coded for maximum content per binit; this infers that the source emits a series of independent binits, each of whose two values (usually 0 and 1) are equally probable.

6.5.2 Symmetric Memoryless Channel

The most efficient modulation system is the phase-reversal keyed; for such a system, the transmission of a 0 or a 1 requires an equal amount of power, and maximum transmission rate (and minimum average probability of error) is obtained when the receiver decision system is adjusted for equal transitional probabilities, 0-transmitted to 1-received, and

1-transmitted to 0-received. A similar situation occurs with all symmetric modulation systems.

By memoryless channel, it is implied that there is no intersymbol interference. The solution for the error rate of a channel having symbol smearing is, for all practical purposes, an unsolved problem; treatment of this situation is beyond the intended scope of this chapter.

6.5.3 Additive White Gaussian Noise

There are two main motivations behind the assumption of additive Gaussian channel perturbance. The first is a practical one, from the viewpoint of analysis; such an assumption greatly facilitates the analysis of system behavior. Greater justification, however, is provided by consideration of the type of system for which error correcting codes hold the greatest benefits. As mentioned in 6.4.1.1, coders are easily implemented, can be made light in weight, and draw little power; decoders, however, can be extremely complex. One of the most critical applications of communication links, so far as minimizing transmitter weight and power requirements while maintaining high information rates and low error rates are concerned, is transmission from space vehicles and satellites to ground stations. In the discussion of channel characterization of Chapter II, it is pointed out (2.3.8) that the frequencies of value for space communications lie above 100 mc. It is further advanced, in 2.2.5, that the majority of the additive disturbances in the 30 to 150 mc range - indeed, virtually all such disturbances, for frequencies above 150 mc - are in fact Gaussian in nature.

6.6 Reliability of Symmetric Mode Binary Modulation Systems

6.6.1 Introduction

The formulae and relationships quoted in this section are derived and/or collated by Hancock and Sheppard in a previous report, "Information

Efficiency of Binary Communications Systems", Contract AF 33(616)-8283. They are presented here only in the interest of providing an analytic basis for the graphical presentation to follow.

6.6.2 PSK/MF - Coherent Detection

This system represents the best possible binary system attainable, with respect to probability of error. The graphical results to follow are based upon this system.

The filter output is described by the conditional probabilities

$$p(x \mid 0_t) = \frac{1}{2\pi N_o E} \, e^{-\frac{(x+E)^2}{2N_o E}} \tag{6-1}$$

and

$$p(x \mid 1_t) = \frac{1}{2\pi N_o E} \, e^{-\frac{(x-E)^2}{2N_o E}} \tag{6-2}$$

where x = filter output

E = energy per binit

N_o = noise spectral density (double-sided)

For symmetric operation, the resulting probability of error is

$$P_e = \frac{1}{2}\left[1 - \text{erf}\left(\sqrt{\frac{E}{2N_o}}\right)\right] \tag{6-3}$$

6.6.3 Summary of Other Systems

ASK - LED: $P_e = e^{-\lambda}$, where λ is the solution to the integral equation (6-4)

$$\int_o^{\lambda} e^{-\left[\alpha + \frac{E}{2N_o}\right]} I_o\left(\sqrt{\frac{2\alpha E}{N_o}}\right) d\alpha = e^{-\lambda} \tag{6-5}$$

ASK - Coherent Detection:

$$P_e = \frac{1}{2}\left[1 - \text{erf}\left(\sqrt{\frac{E}{8N_o}}\right)\right] \tag{6-6}$$

PSK - Synchronous Detection:

$$P_e = \frac{1}{2}\left[1 - \text{erf}\left(\sqrt{\frac{E}{2N_o}}\right)\right] \tag{6-7}$$

PSK - Phase Comparison:

$$P_e = \frac{1}{2} e^{-\frac{E}{4N_o}} \tag{6-8}$$

ASK/MF - LED: $P_e = e^{-\lambda}$, where λ is the solution to (6-9)

$$\int_o^{\lambda} e^{-\left[\alpha + \frac{E}{N_o}\right]} I_o\left(\sqrt{\frac{4\alpha E}{N_o}}\right) = e^{-\lambda} \tag{6-10}$$

ASK/MF - Coherent Detection:

$$P_e = \frac{1}{2}\left[1 - \text{erf}\left(\sqrt{\frac{E}{4N_o}}\right)\right] \tag{6-11}$$

FSK/MF - LED:

$$P_e = \frac{1}{2} e^{-\frac{E}{4N_o}} \tag{6-12}$$

FSK/MF - Coherent Detection:

$$P_e = \frac{1}{2}\left[1 - \text{erf}\left(\sqrt{\frac{E}{4N_o}}\right)\right] \tag{6-13}$$

PSK/MF - Phase Comparison:

$$P_e = \frac{1}{2} e^{-\frac{E}{N_o}} \tag{6-14}$$

6.7 Performance of Binary EC/ED Codes

6.7.1 Introduction

This section is concerned specifically with the derivation, analytically and/or experimentally, of what is termed the "error rate equation".

The error rate equation is defined to be the equation for the binit probability of error at the decoder output given, as the independent variable, the channel word or digit error probability.

It is assumed throughout that the transitional probabilities for the channel are equal, and that the probability of a single error in the channel is independent of the past history of the channel.

It should be noted that errors at the output of the decoder no longer occur independently. All simple error patterns received by the decoder are corrected, while those of higher order are not; hence, the output errors occur in bursts.

6.7.2 General Error Rate Equation

In the derivation of an error rate equation, the first logical step is to express the decoder output error probability P_e' as a summation:

$$P_e' = \sum_{\substack{\text{all input}\\ \text{error patterns}}} \begin{array}{l} \text{P(arbitrary information binit in error at the} \\ \quad \text{decoder output} \mid \text{the specific input error pattern)} \\ \text{P(a specific input error pattern)} \end{array} \qquad (6\text{-}15)$$

For a symmetric channel with independent errors and error probability p, and for group codes,

$$\text{P\{specific input error pattern\}} = P_e^i \, (1-P_e)^{n-i} \qquad (6\text{-}16)$$

where

n = word length

i = number of errors in the pattern

since each word is decoded independent of the other words received.

P_e' may also be expressed as

P_e' = Probability that an arbitrary information binit is in error

$$= \sum_{\substack{\text{all info}\\ \text{binits in the}\\ \text{code word}}} \begin{array}{l} \text{P\{an arbitrarily chosen info binit = a specific info} \\ \quad \text{binit in the code word\}} \\ \text{P\{the specific info binit in the previous condition is} \\ \quad \text{in error at the decoding output\}} \end{array} \qquad (6\text{-}17)$$

Define k = number of information binits in the code word. Arrange these binits in a sequence so that "the αth binit", reads as "the αth binit in the sequence of k information binits in a code word", refers to a unique binit.

Now, P an arbitrarily chosen info binit = a specific info binit = $\frac{1}{k}$ (6-18)

thus,

$$P_e' = \frac{1}{k} \sum_{\alpha=1}^{k} P\{\text{the } \alpha\text{th info binit at the decoder output is in error}\} \tag{6-19}$$

Consider the set of all binits in the code word; with each binit associate a number d_j, $1 \leq d_j \leq n$, so that by referring to "the d_j^{th} binit", reference is made to a unique binit in the word.

Every information binit is also in the set; let d_α = the code word binit corresponding to the α^{th} information binit, as previously defined. Then,

$$P_e' = \frac{1}{k} \sum_{\alpha=1}^{k} P\{d_\alpha \text{ is in error at the decoder output}\} \tag{6-20}$$

Define (e_j') as the specific set of binits in a word in error at the decoder output, with $1 \leq j \leq i'$, i' = total number of errors in the word at the decoder output. Then each e_j' corresponds to a d in error.

Define (e_j) in a similar manner, <u>but for the set of errors at the decoder input resulting in the set (e_j') at the output</u>. Here, $1 \leq j \leq i$, and i is not generally the same as i'. Then, with "ϵ" read as "belongs to" or "is included in",

$$P_e' = \frac{1}{k} \sum_{\alpha=1}^{k} \sum_{\text{all } (e_j)} P\{d_\alpha \epsilon (e_j')\} \, P\{(e_j)\} \tag{6-21}$$

Now, $P\{(e_j)\} = P_e^i (1-P_e)^{n-i}$ (6-22)

Note, however, that $P\{d_\alpha \epsilon (e_j')\}$ is, <u>for a given (e_j'),</u> such that either $d_\alpha \epsilon (e_j')$ and $P\{d_\alpha \epsilon (e_j')\} = 1$, or $d_\alpha \notin (e_j')$ and $P\{d_\alpha \epsilon (e_j')\} = 0$.

Define $N_{i\alpha}$ = the number of received error patterns (e_j) containing i errors each for which $d_\alpha \epsilon (e_j')$. Then

$$P_e' = \frac{1}{k} \sum_{\alpha=1}^{k} \sum_{i=1}^{n} N_{i\alpha} P_e^{\,i} (1-P_e)^{n-i} \qquad (6$$

The problem is now one of determining the parameters $N_{i\alpha}$.

6.7.3 Specific Solution Methods

6.7.3.1 Computer Simulation

Rewrite Eq. (6-22), thus

$$P_e' = \sum_{i=1}^{n} \frac{1}{k} \left[\sum_{\alpha=1}^{k} N_{i\alpha} \right] P_e^{\,i} (1-P_e)^{n-i} \qquad (6$$

Now, for error correcting codes, $N_{i\alpha}$ = the number of received error patterns (e_j) containing i errors for which the associated d_α is in error; then $\sum_{\alpha=1}^{k} N_{i\alpha}$ is just the total number of information binits in error at the decoder output as a result of <u>all</u> of the i-fold error pattern inputs, and $\frac{1}{k} \sum_{\alpha=1}^{k} N_{i\alpha}$ is the average number of times an information binit is in error as a result of all $\binom{n}{i}$ i-fold input error patterns; i.e., then,

$$\frac{\frac{1}{k} \sum_{\alpha=1}^{k} N_{i\alpha}}{\binom{n}{i}} = \text{the probability that an arbitrary information binit is in error}$$

given that some i-fold error pattern occurred at the decoder input

and

$$\binom{n}{i} P_e^{\,i} (1-P_e)^{n-i} = \text{probability of an i-fold error pattern input.}$$

Returning to (6-23), a method of solution by computer simulation is obvious. Set up a decoder on the computer and, with an assumed "transmitted" all-zero code word, simulate all possible error patterns (by generating all 2^n n binit binary numbers) and apply these to the decoder. Then for each value of i ones (i.e., errors) in a code word, record the total number of ones (errors) in the information binits at the decoder output for all such i-fold patterns. This number is, then, $\sum_{\alpha=1}^{k} N_{i\alpha}$.

This method of solution, although straightforward, is quite lengthy. For a code of length n, the number of error combinations that must be examined is 2^n -- and the method of examination (i.e., the decoding process) can be quite complex. Analytic solution, where possible, is preferred.

6.7.3.2 Analytic Approach; Hamming SEC Codes

Several simplifications are possible when dealing with Hamming codes; these arise, basically, as a result of these codes being perfect. (Although this property is not used explicitly, the results implied by this property are invaluable).

The first simplifying property is the relationships between the (e_j) and (e_j'). For any received error patterns (e_j), the "corrected" error pattern (e_j') must fall into one of three categories: it is identical with (e_j); it is (e_j) with one error deleted; or, it is (e_j) with one error added.

Secondly, it is possible to show (see Appendix IV) that the number of error patterns (e_j) of fixed length i for which the corresponding (e_j') is such that $(e_j') = (e_j)$ with β adjoined $\beta \notin (e_j)$, for some fixed β, is independent of the value of β considered; a similar condition exists for all (e_j') formed by deleting β from an (e_j) of length i (and here $\beta \in (e_j)$ is implied).

It is also proved in Appendix IV that the number of (e_j) of length i for which the associated $(e_j') = (e_j)$ and $\alpha\epsilon(e_j')$; for which $(e_j') = (e_j)$ with some $\beta \not\epsilon (e_j)$ adjoined, and $\alpha\epsilon(e_j')$; and for which $(e_j') = (e_j)$ with some β deleted $\beta\epsilon(e_j)$, and $\alpha\epsilon(e_j')$; are each independent of the α chosen. From these, it is obvious that the $N_{i\alpha}$ are independent of α; redefining $A_i = N_{i\alpha}$, (6-23) becomes

$$P_e' = \sum_{i=1}^{n} A_i P_e^{\,i}(1-P_e)^{n-i} \qquad (6\text{-}25)$$

and

A_i = the number of received error patterns (e_j) of weight (number of errors) i for which the "corrected" error patterns (e_j') contain some specific binit chosen from the full code word.

6.7.3.3 Analytic Approach: Hamming SEC/DED Codes

For these codes, the original definition of P_e' must be examined. In this report, it is assumed that those error patterns of order large enough to be detected but not corrected are to result in the entire word being discarded -- i.e., the complete lack of reception is preferable to accepting as valid a group of information binits known to contain large numbers of errors. With reasonably small channel error probabilities, the average number of words discarded is shown to be an extremely small fraction of the total received words, while the multiplicative increase in reliability is of the order of 10 to 100, compared to the SEC codes.

Then, P_e' = Probability of an arbitrary information binit being in error after decoding, given that the word in which the binit was contained was not discarded.

In Appendix IV, the indicator y is defined as having a value 1 for words that are not discarded, and 0 for those that are. Then

$$P_e' = P\{\text{arbitrary information binit in error after decoding} \mid y = 1\} \quad (6\text{-}26)$$

The actual analysis and the resulting computations are simplified by working with the formulation

$$P_e' = \frac{P\{\text{arbitrary information binit in error after decoding } \underline{\text{and}}\ y = 1\}}{P\{y = 1\}} \quad (6\text{-}27)$$

The numerator may then be expanded as discussed previously, and

P{ arbitrary information binit in error after decoding <u>and</u> y = 1}

$$= \sum_{i=1}^{n'} \left[\frac{1}{k} \sum_{\alpha=1}^{k} N_{i\alpha}' \right] P_e^{\,i}(1-P_e)^{n'-i} \quad (6\text{-}28)$$

with $n' = 2^m$ = code word length $(= n + 1)$ and $N_{i\alpha}'$ = the number of received error patterns of weight i for which y = 1 <u>and</u> the associated d is in error.

Two different conditions for discarding the received word are studied. The first of these, and the more common, is that the overall parity check is satisfied, but the internal checks are not -- this corresponds to the number of received errors being even, and $(e_j') \neq (e_j)$. This discards all double errors, as well as most even weight error patterns.

The second condition considered is that for which the criterion of the first applies and, alternatively, the condition that the informal parity checks are satisfied while the overall check is not. This then detects and discards many of the odd-weight error patterns as well; unfortunately, it also discards the one weight=1 pattern for which the error occurs in the overall check binit.

For a Hamming SEC/DED code operating under the first condition of word-discard, the errors patterns of interest (for even i) are those for which (e_j) is such that $(e_j') = (e_j)$, length = i, and length = i-1 (i-1 corresponding to the i-weight error patterns with one of the errors in the overall check binit). As previously discussed, the relationships required are shown in Appendix IV, to be independent of the particular α under consideration.

6.7.4 Summary of Hamming Code Error Rate Equations

The following results are derived in detail in Appendix IV.

For the SEC codes of length $n = 2^m-1$,

$$P_e' = \sum_{i=o}^{n} \frac{1}{n} [(i-1) M_i + iN_i + (i+1) L_i] P_e^{\,i}(1-P_e)^{n-i}, \tag{6-29}$$

where M_i = number of error patterns of weight i for which (e_j') has weight i-1 (i.e., deletion of some member of (e_j) to form (e_j'));

N_i = number of error patterns of weight i for which $(e_j') = (e_j)$;

L_i = number of error patterns of weight i for which (e_j') has weight (i + 1) (i.e., adjoining some $\beta \notin (e_j)$ to (e_j) to form (e_j')).

Note that this form preserves the physical meaning of the parameters. With M_i, N_i and L_i as defined above, the probability of receiving a word of error-pattern weight i satisfying the conditions defined for M_i is just $M_i P_e^i(1-P_e)^{n-i}$ <u>If the assumption is made that the probability of error for a binit after "correction" is independent of that binit being an information binit</u>, then the probability of an arbitrary information binit being in error is just

$$P_e' = \sum_{\substack{\text{weights} \\ \text{of } (e_j')}} \frac{\text{weight of } (e_j)}{n} \cdot P\{(e_j') \text{ having the given weight}\} \qquad (6\text{-}30)$$

The probability of (e_j') having weight i is just

$$L_{i-1}P_e^{i-1}(1-P_e)^{n-i+1}+N_iP_e^i(1-P_e)^{n-i}+M_{i+1}P_e^{i+1}(1-P_e)^{n-i-1} ,$$

and P_e' becomes

$$P_e' = \sum_{i=o}^{n} \frac{i}{n}\left[L_{i-1}P_e^{i-1}(1-P_e)^{n-i+1}+N_iP_e^i(1-P_e)^{n-i}+M_{i+1}P_e^{i+1}(1-P_e)^{n-i-1}\right] \qquad (6\text{-}31)$$

-- now, $L_{-1} = 0$ and $M_{n+1} = 0$, obviously. Thus

$$P_e' = \sum_{i=o}^{n} \left[\frac{i+1}{n} L_i\, P_e^i(1-P_e)^{n-i}+ \frac{i}{n} N_iP_e^i(1-P_e)^{n-i}+ \frac{i-1}{n} M_i\, P_e^i(1-P_e)^{n-i}\right], \qquad (6\text{-}32)$$

as before.

It is shown in Appendix IV that the parameters L, M and N are related by the iterative equations,

$$\begin{aligned} M_i &= (n-i+1)N_{i-1} \\ N_i &= \frac{1}{i} L_{i-1} \\ L_i &= \binom{n}{i} - N_i - M_i \end{aligned} \qquad (6\text{-}33)$$

with initial values $M_o = L_o = 0$; $N_o = 1$

For the SEC/DED codes of length $n' = n+1 = 2^m$, operating under the first word-discard conditions discussed above,

$$P_e' = \frac{1}{P\{y=1\}} \left\{ \sum_{\substack{i=2 \\ (i\ \text{even})}}^{n+1} \frac{1}{n} \left[iN_i + (i-1)N_{i-1} \right] P_e^i (1-P_e)^{n+1-i} \right.$$

$$\left. + \sum_{\substack{i=1 \\ (i\ \text{odd})}}^{n} \frac{1}{n} \left[(i-2)M_{i-1} + (i-1)M_i + (i-1)N_{i-1} + iN_i + iL_{i-1} + (i+1)L_i \right] P_e^i (1-P_e)^{n+1-i} \right\} \tag{6-34}$$

and

$$P\{y=1\} = \sum_{\substack{i=o \\ (i\ \text{even})}}^{n+1} \left[N_i + N_{i-1} \right] P_e^i (1-P_e)^{n+1-i} + \sum_{\substack{i=1 \\ (i\ \text{odd})}}^{n} \binom{n+1}{i} P_e^i (1-P_e)^{n+1-i} \tag{6-35}$$

-- the "physical interpretation" analysis, along the lines of that for equation (6-30), is obvious.

The condition for discarding the word in this case is that $(e_j') \neq (e_j)$ <u>and</u> i = even.

For the second set of word-discard conditions,

$$P_e' = \frac{1}{P\{y=1\}} \left\{ \sum_{\substack{i=2 \\ (i\ \text{even})}}^{n+1} \frac{1}{n} \left[iN_i + (i-1)N_{i-1} \right] P_e^i (1-P_e)^{n+1-i} \right.$$

$$\left. + \sum_{\substack{i=1 \\ (i\ \text{odd})}}^{n} \frac{1}{n} \left[(i-2)M_{i-1} + (i-1)M_i + iL_{i-1} + (i+1)L_i \right] P_e^i (1-P_e)^{n+1-i} \right\} \tag{6-36}$$

with

$$P\{y=1\} = \sum_{\substack{i=o \\ (i\ \text{even})}}^{n+1} \left[N_i + N_{i-1} \right] P_e^i (1-P_e)^{n+1-i} + \sum_{\substack{i=1 \\ (i\ \text{odd})}}^{n} \left[M_{i-1} + M_i + L_{i-1} + L_i \right] P_e^i (1-P_e)^{n+1-i} \tag{6-37}$$

with the conditions for discarding a word being $(e_j') \neq (e_j)$ <u>and</u> i = even, <u>or</u> $(e_j') = (e_j)$ <u>and</u> i = odd.

In all cases, for $i < 0$ or $i > n$, $L_i = M_i = N_i = 0$.

6.8 Results

This section contains detailed numerical analyses of the performance of Hamming SEC codes of lengths 7, 15, 31, 63, 127, 255 and 511 binits; SEC/DED codes of lengths 8, 16, 32, 64, 128, 256 and 512 binits; and the Bose-Chandhuri (15, 5) and (15, 7) codes; in all cases, the modulation-detection system used is phase-shift keying with matched filter reception. A comparison and conversion graph is supplied for use with other symmetric systems. A brief introduction to each subsection, with examples of the use of the graphs, is included.

During the compilation of results, it was found that the probabilities of error for the SEC/DED codes operating under the second set of word-discard conditions was only marginally better than those for such codes operating under the first, more common set, while the probability of word-discard was greatly increased. For this reason, numerical results for the second set of conditions have been omitted.

6.8.1 Fixed Bandwidth Analysis

The graphs included in Figs. 6.1 through 6.12 are based upon a fixed bandwidth restriction - i.e., the information binit rate of the coded system is reduced in proportion to the redundancy of the code, maintaining a constant transmitter rate.

Table 6-1 lists the information binit rate of each coded system, based upon an uncoded rate of unity. For low error probability, this

rate is very nearly the information rate.

As an example, consider a PSK-MF system for which the Ratio $\frac{E}{N_o}$ is 9.5 db. The bandwidth of the channel is fixed; however, a reduction in information binit transmission of 8% is permitted. What decrease in error probability is attainable?

From Table 6-1, the shortest Hamming code that can be used is either the (127, 120) SEC code, or, if binit rejections are permitted, the (128, 120) SEC/DED code. The uncoded error probability is 1.4×10^{-3}; (Fig. 6.3). With the SEC/DED code, the factor is 52, reducing the error rate to 2.7×10^{-5} (Fig. 6.7), but this introduces information binit rejections by the receiver with a probability of 1.2×10^{-2} (Fig. 6.11).

Code	Rate	Code	Rate
Hamming -------------SEC		Hamming ------------SEC/DED	
(7, 4)	0.571	(8, 4)	0.500
(15, 11)	0.733	(16, 11)	0.683
(31, 26)	0.839	(32, 36)	0.808
(63, 57)	0.905	(64, 57)	0.891
(127, 120)	0.945	(128, 120)	0.938
(255, 247)	0.969	(256, 247)	0.965
(511, 502)	0.982	(512, 502)	0.980
Bose-Chandhuri Codes			
(15, 7)	0.467	(15, 5)	0.333

TABLE 6-1

INFORMATION BINIT RATE-FIXED BANDWIDTH SYSTEM

(No Coding = 1)

6.8.2 Fixed Information Binit Rate

Figs. 6.13 through 6.24 provide a code performance analysis under the restriction of constant rate of information binit transmission. To provide a criterion of comparison, the ordinate of the graphs is the ratio $\frac{E'}{N_o}$, in db, where E' is the energy per information binit (for the uncoded case, this is then the energy per transmitted binit).

In this case, then, the actual energy per transmitted binit is reduced from the graphed value in proportion to the redundancy of the code under consideration.

Example: A PSK-MF system is to be used under a transmitted-power restriction that results in an uncoded $\frac{E}{N_o}$ of 11.0 db. If the information binit transmission rate is to be maintained, what is the shortest Hamming SEC code that will result in an output error probability of 10^{-4}?

To hold both the information binit transmission rate and the average power constant, the energy per information binit must be held fixed - i.e., $\frac{E'}{N_o}$ is to remain at 9.0 db. Reference to Figs. 6.13 through 6.16 shows that the shortest code that satisfies the error rate requirement is the (15, 11) code and this results in an error rate of 3.5×10^{-5}.

An interesting phenomenom is emphasized by the constant information binit rate graphs -- that the "best" code, in terms of lowest probability of error, is not always either the shortest or the longest code permissible. The longer codes lose less power due to redundancy, but have greater inherent error rates, while the words of the shorter codes, although inherently less prone to multiple errors, sacrifice much of the transmitter power in the check binit transmission. Generally, then, at

any fixed power level, constant information binit rate operation will result in an optimum code in a particular set of codes.

In particular, for all of the Hamming SEC codes investigated, no improvement at all is possible below an $\frac{E'}{N_o}$ ratio of 7.0 db. From 7.0 db to 10.1 db, the (31, 26) code results in the lowest P_e' ; from 10.1 to approximately 11.2 db, the (63, 57) code is best, while from 11.2 db to approximately 15 db, the (127, 120) code is optimum. From 15 db out to the maximum ratio studied, both the (255, 247) and the (511, 502) codes give approximately equal, and lowest error probabilities. A comparable situation exists for the SEC/DED codes.

An interesting feature of the SEC/DED codes is that the probability of rejection is asymptotic to 0.5 as the $\frac{E}{N_o}$ ratio drops. This is a natural outcome of the fact that, for high channel error probabilities, the probability of a received word containing at most one error becomes very small; for the remaining error patterns, all of those with even parity (neglecting those for which the check word is zero) are discarded -- i.e., the probability of rejection approaches the probability of an arbitrarily chosen set of binary digits having even parity.

6.8.3 Merit

The merit graphed in Figs. 6.25 through 6.31 is arrived at by calculating the ratio $\frac{E'}{N_o}$ required to obtain a given error rate for the coded system, and dividing this into the corresponding $\frac{E'}{N_o} = \frac{E}{N_o}$ for the uncoded system. The resulting figure indicates 1) the factor by which the transmitted power may be reduced (while maintaining a constant information binit rate) by the use of coding, or 2) the increase in information binit rate attainable at a fixed average transmitter power.

Example: An uncoded PSK-MF system is operating with an error rate of 0.15×10^{-4}. With no restrictions on bandwidth, how much faster may the information be transmitted, with the same average transmitter power and error probability, if a Hamming SEC code with N = 63 is used? If an increase of 30% is desired, how much power can be saved while simultaneously achieving this increase, using this code?

Referring to Fig. 6.26, the merit of the (63, 57) code at $P_e' = 0.15 \times 10^{-4}$ is 1.40; thus, the information binit rate may be increased by this factor. If an increase to 1.30 times the original rate is desired, the average power may be reduced by 1.40/1.30 = 1.08, or 0.3 db.

The existence of an optimum length code for a given error probability/uncoded $\frac{E}{N_o}$ ratio range, when operating under a fixed information binit rate constraint, as discussed in 6.8.2, is again illustrated by the merit graphs. (Recall that these graphs are based upon either the increase in the information binit rate at a constant average power, or, alternatively, the allowable power decrease at constant information binit rate.) Moreover, these graphs expand this information, making a more accurate determination of the crossover points possible. These are listed in Table 6-2.

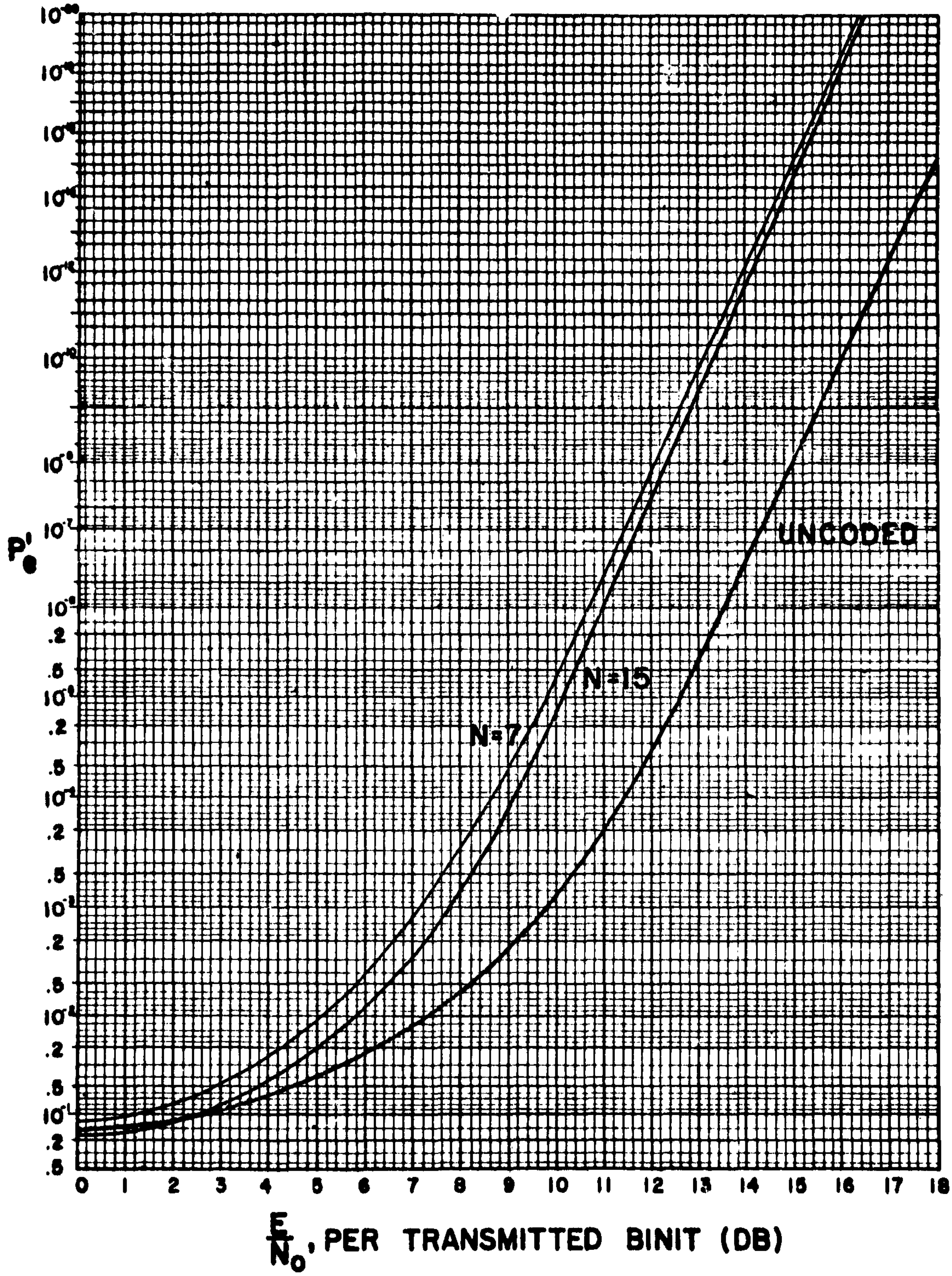
	Error Probability Range	Optimum Length
SEC codes:	above 1.2×10^{-2}	Uncoded
	1.2×10^{-2} to 8.5×10^{-3}	15
	8.5×10^{-3} to 2.0×10^{-4}	31
	2.0×10^{-4} to 3.5×10^{-7}	63
	3.5×10^{-7} to 10^{-12}	127
	10^{-12} to 10^{-18}	255
	10^{-18} to below 10^{-20}	255/511
SEC/DED codes:	above 0.12	Uncoded
	0.12 to 1.4×10^{-2}	8
	1.4×10^{-2} to 6.5×10^{-4}	16
	6.5×10^{-4} to 1.4×10^{-6}	32
	1.4×10^{-6} to 10^{-11}	64
	10^{-11} to 2×10^{-19}	128
	2×10^{-19} to below 10^{-20}	256

TABLE 6-2

OPTIMUM CODE LENGTH PSK/MF SYSTEM

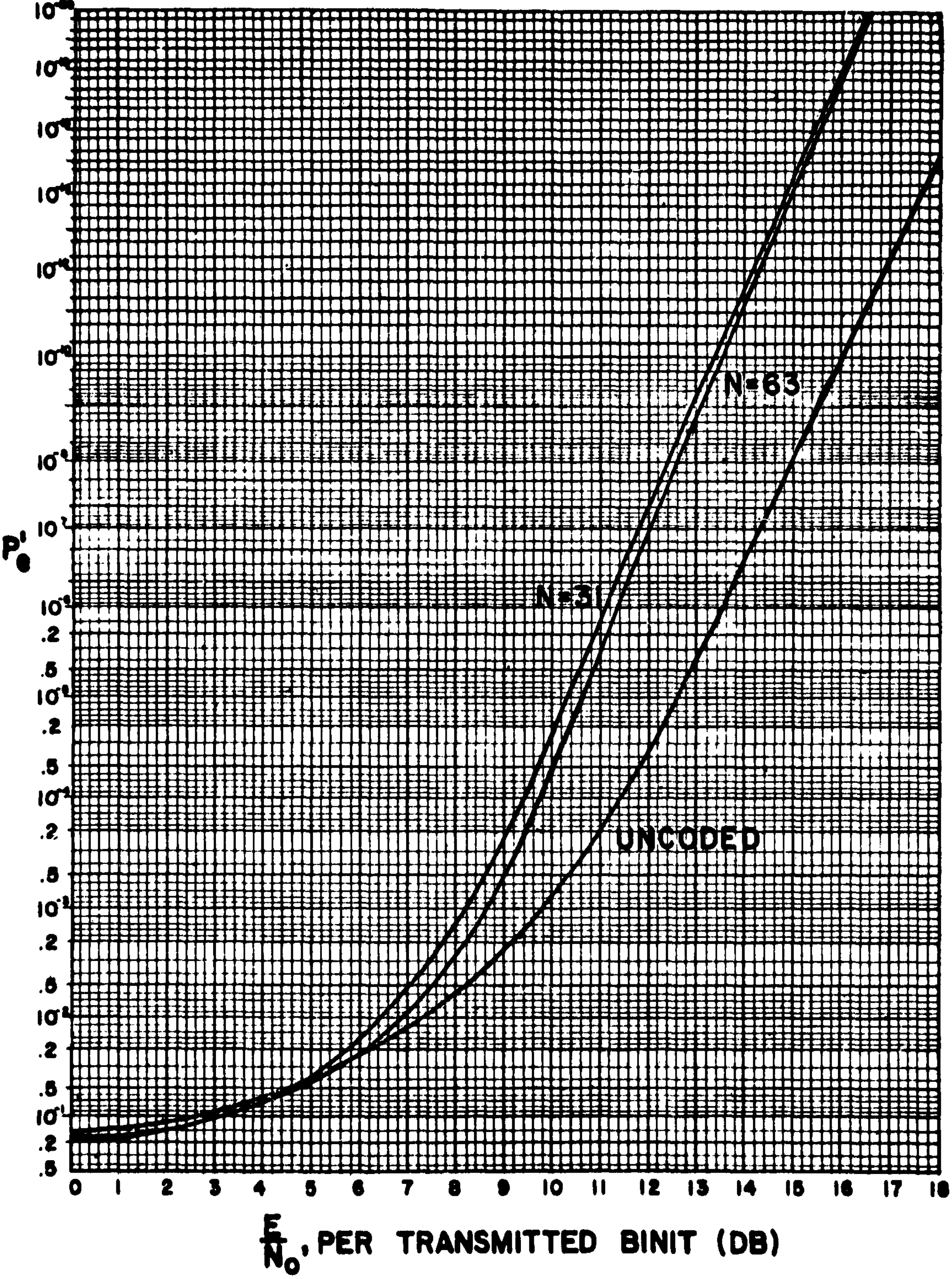
It should be noted that the two Bose-Chandhuri codes analyzed never, in the range for which the merit exceeds unity, out-perform the optimum Hamming code; this is a natural result of the high redundancy of the Bose-Chandhuri codes.

Finally, it must be realized that the increases in information rate permitted by coding are conditional upon the effects of increasing the transmitted binit rate and the system bandwidth, other than the resulting energy-per-binit decrease already considered.



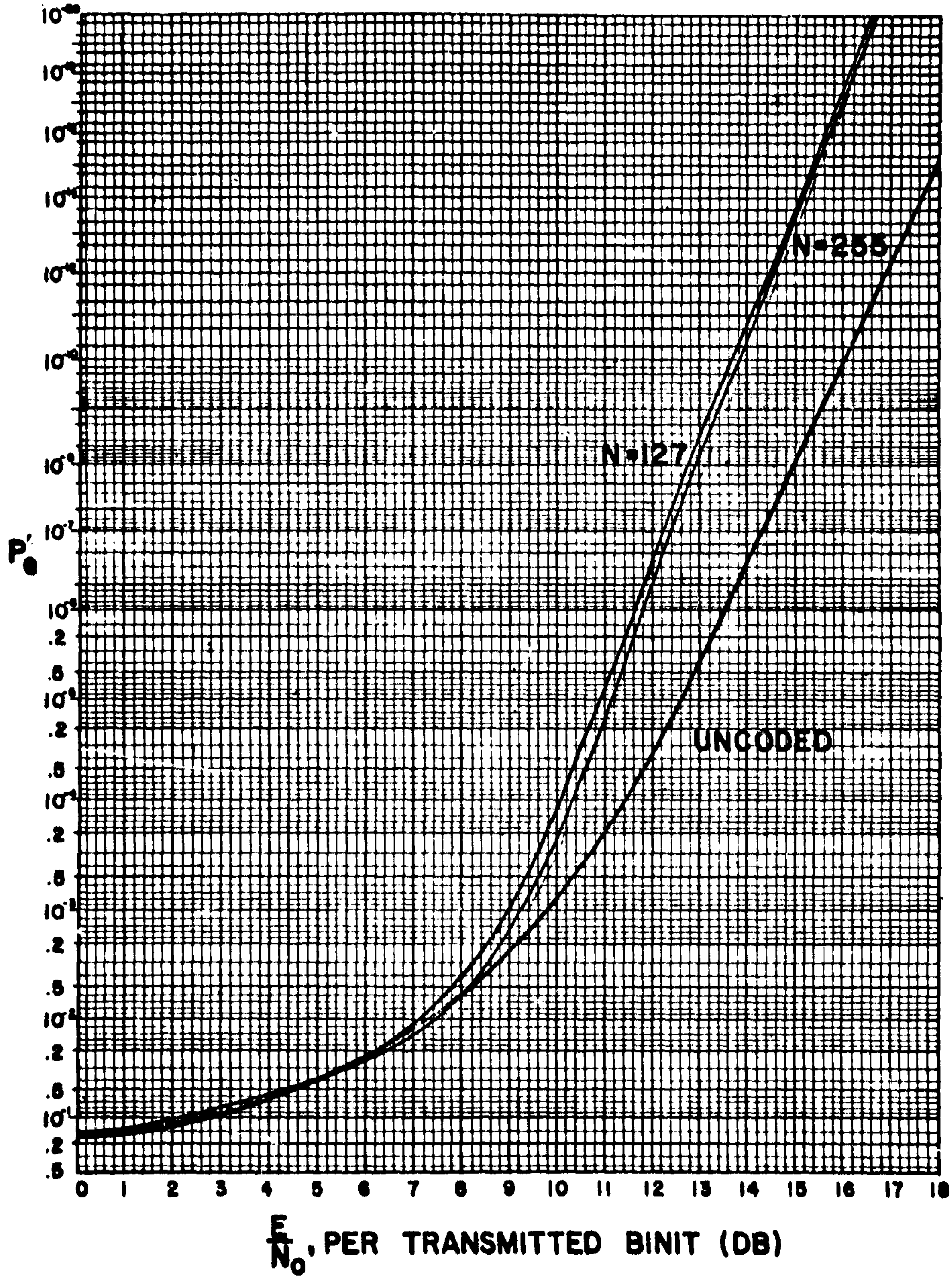
ERROR RATES: HAMMING SEC CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.1



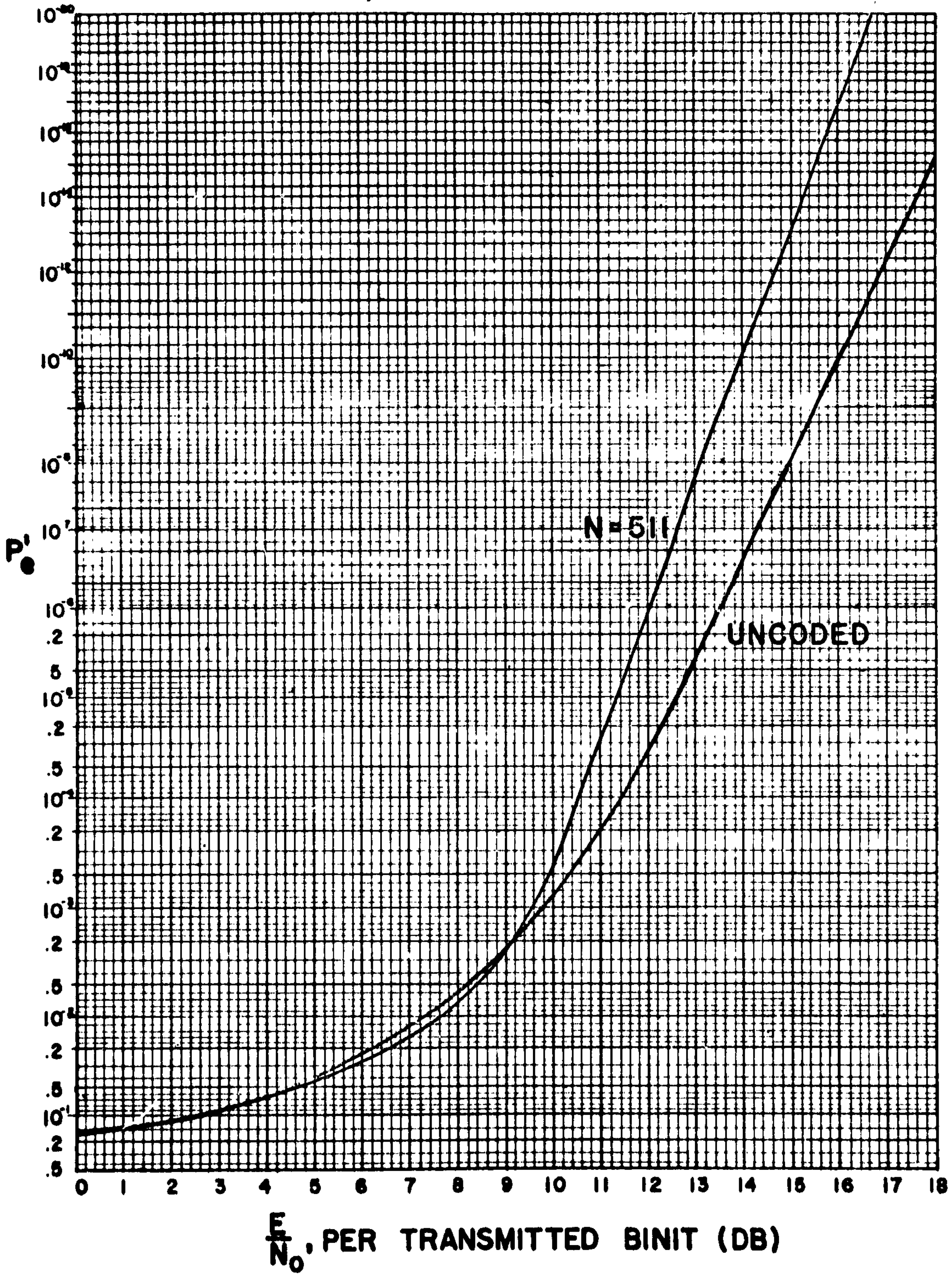
ERROR RATES: HAMMING SEC CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.2



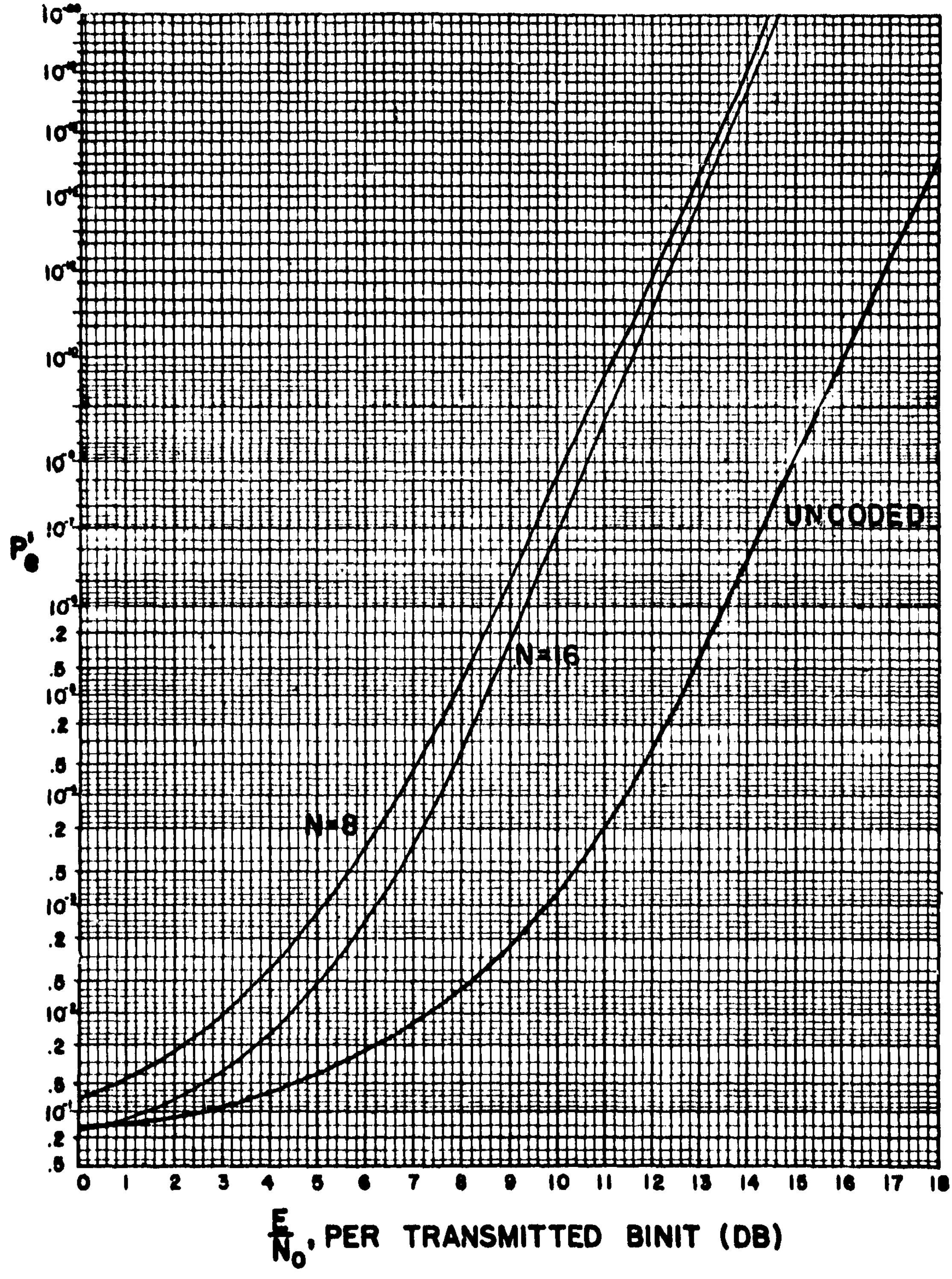
ERROR RATES: HAMMING SEC CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.3



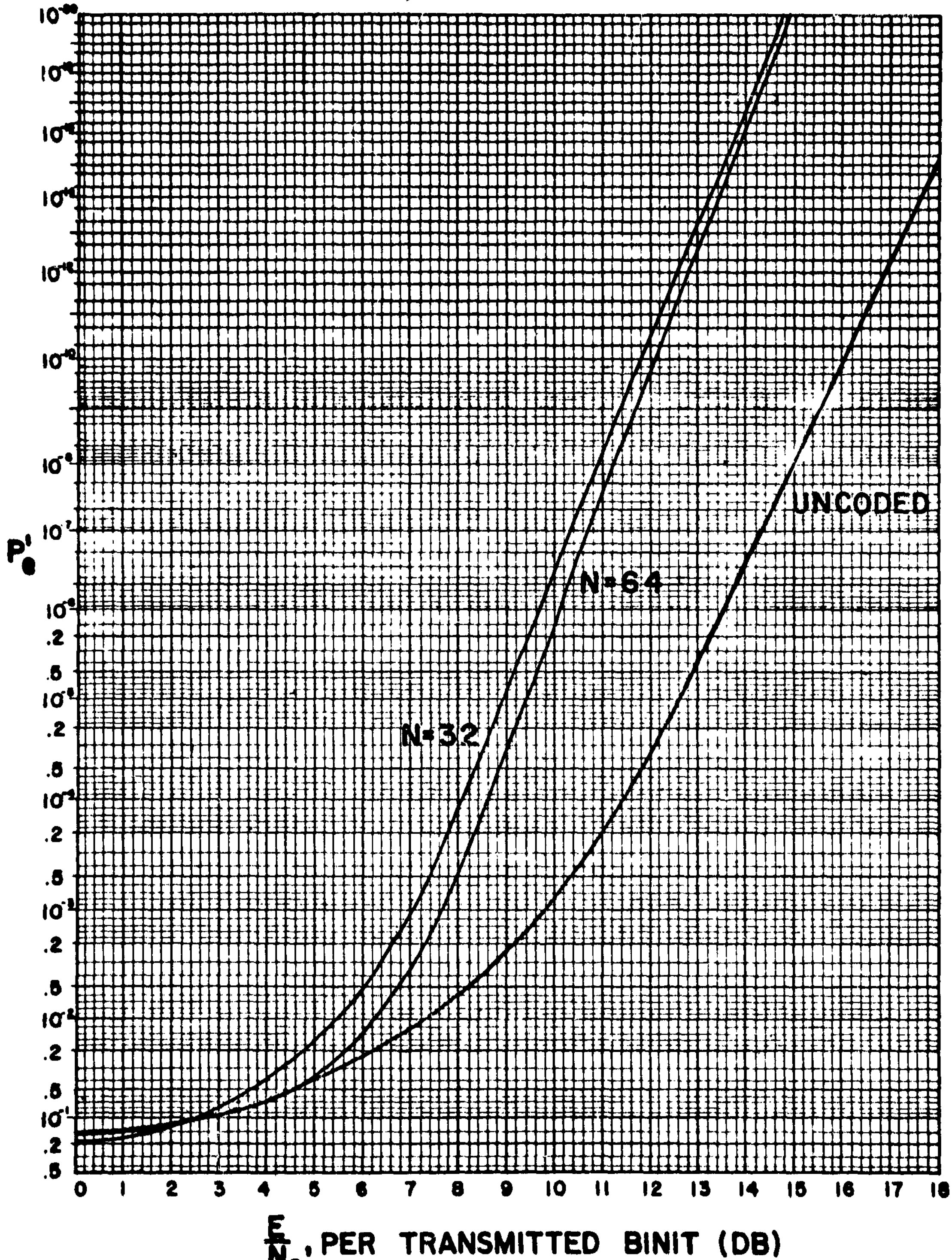
ERROR RATES: HAMMING SEC CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.4



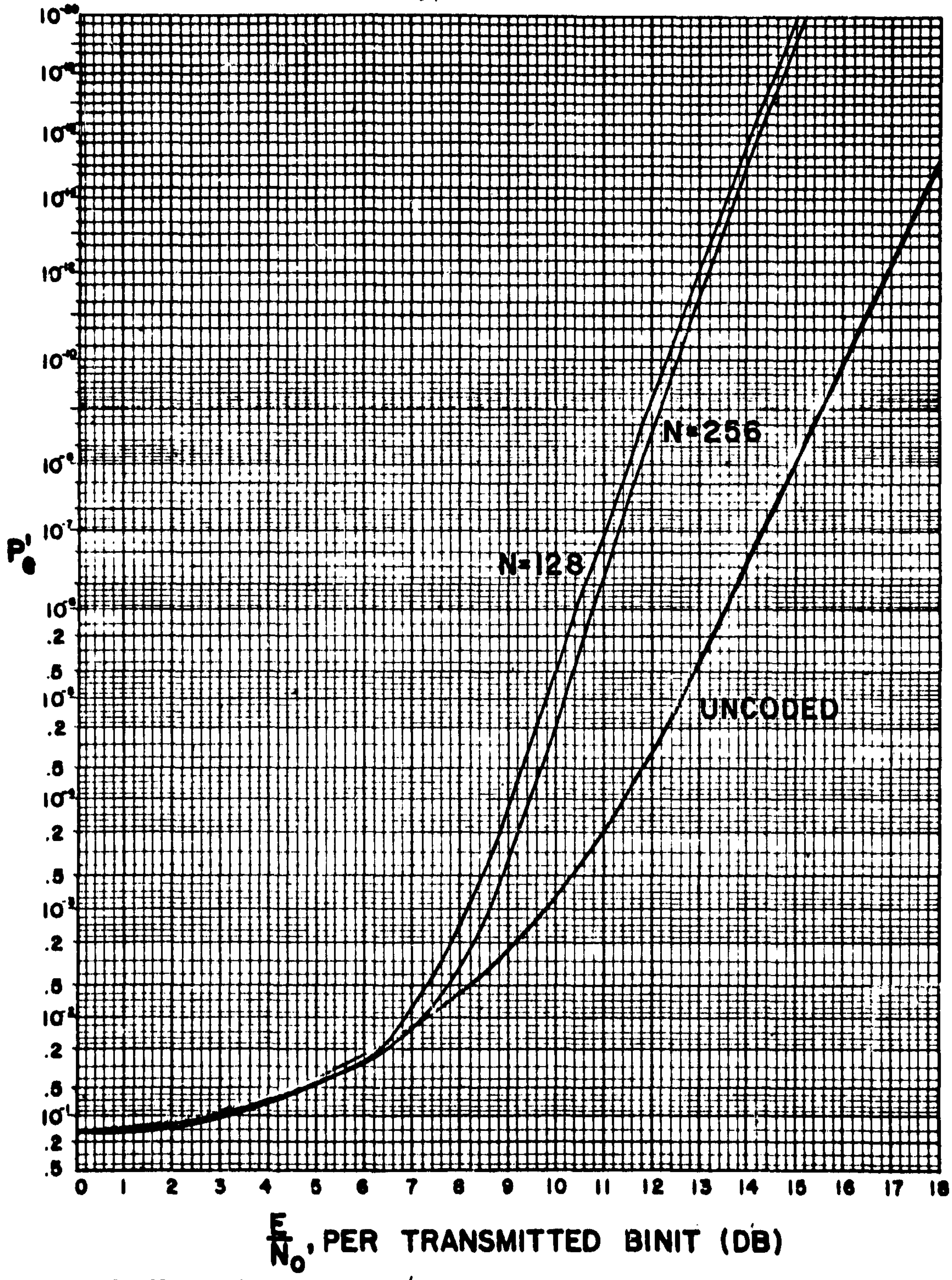
ERROR RATES: HAMMING SEC/DED CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.5



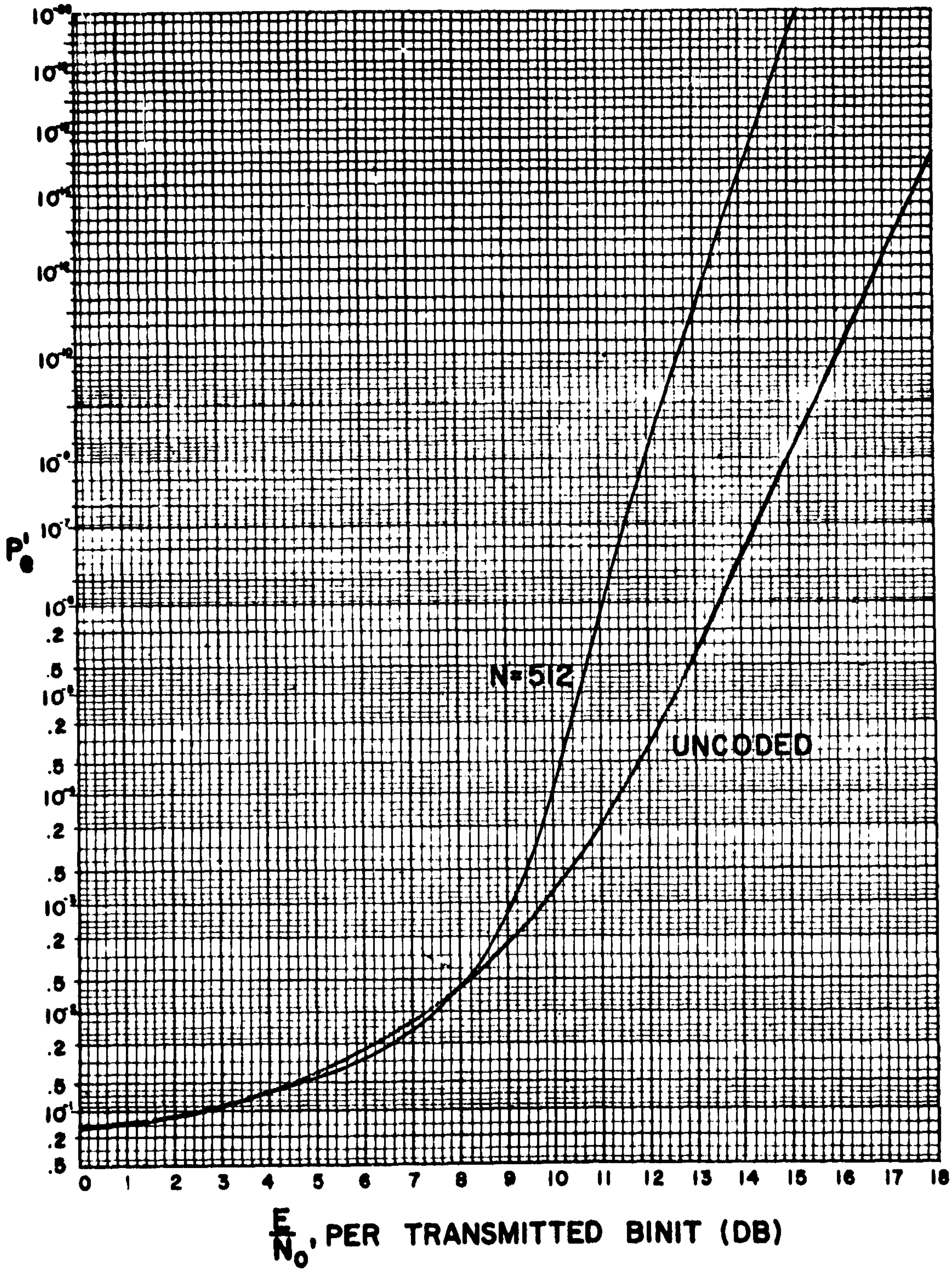
ERROR RATES: HAMMING SEC/DED CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.6



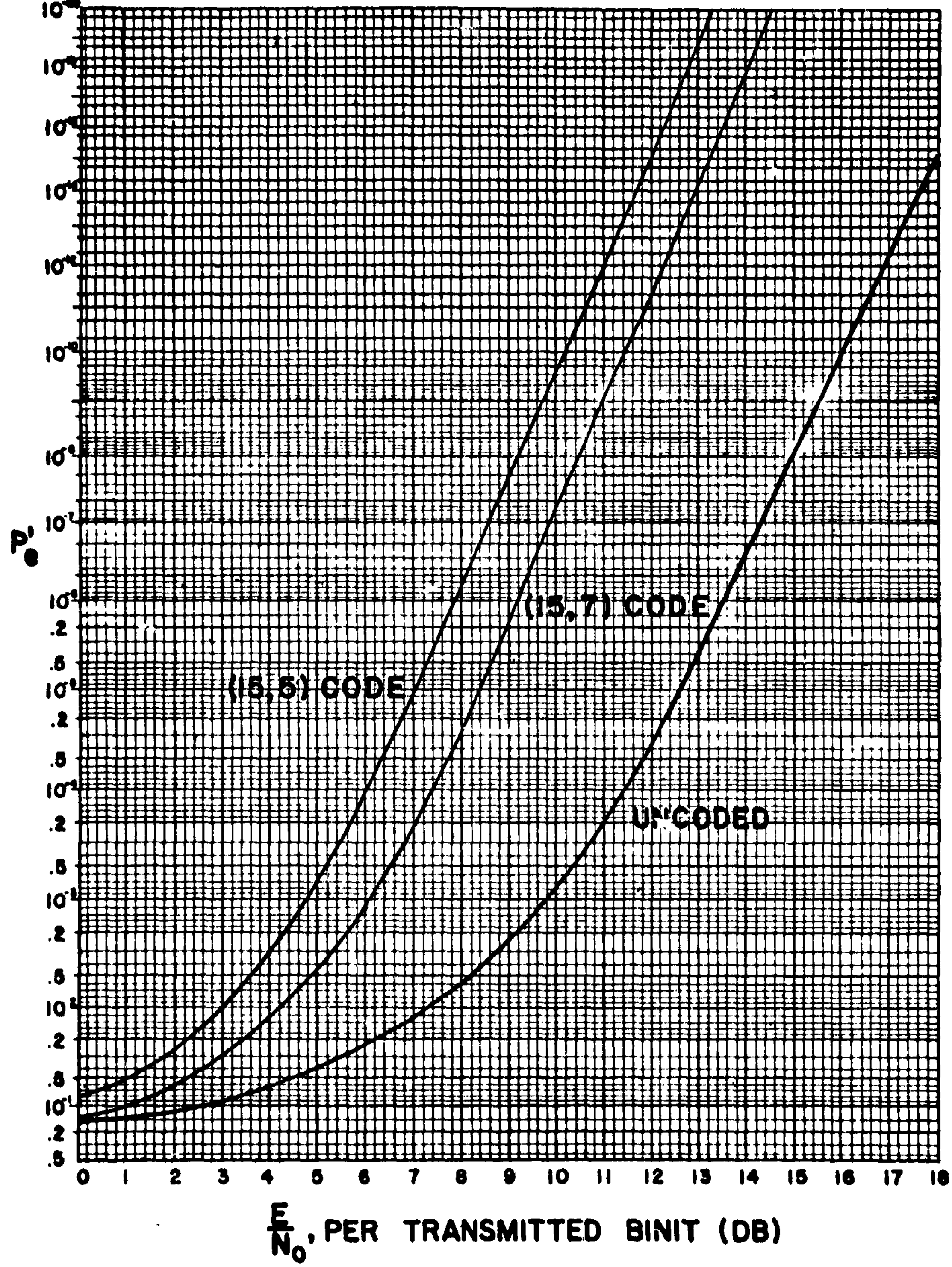
ERROR RATES: HAMMING SEC/DED CODES -- FIXED BANDWIDTH SYSTEM

FIGURE 6.7



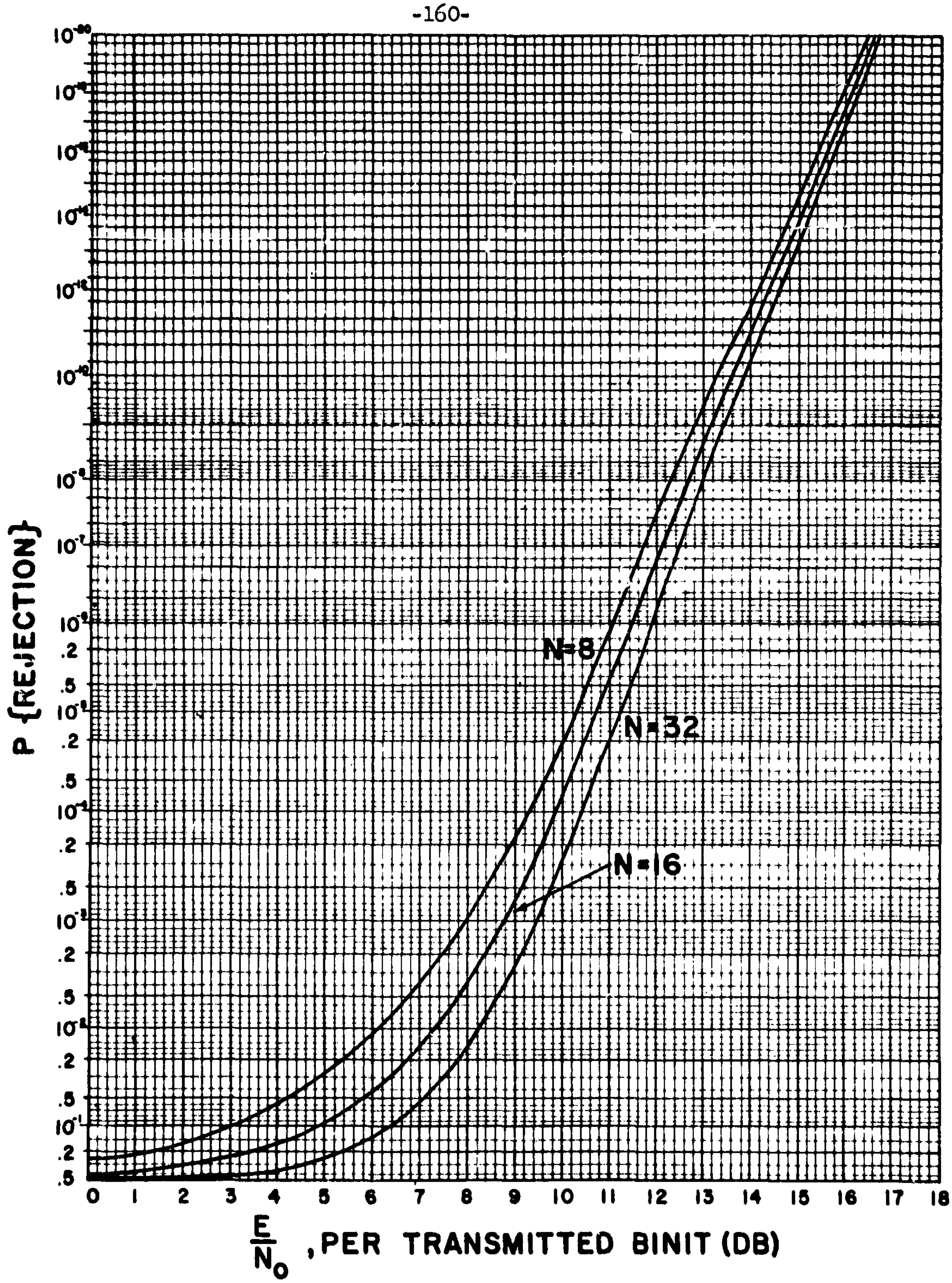
ERROR RATES: HAMMING SEC/DED CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.8



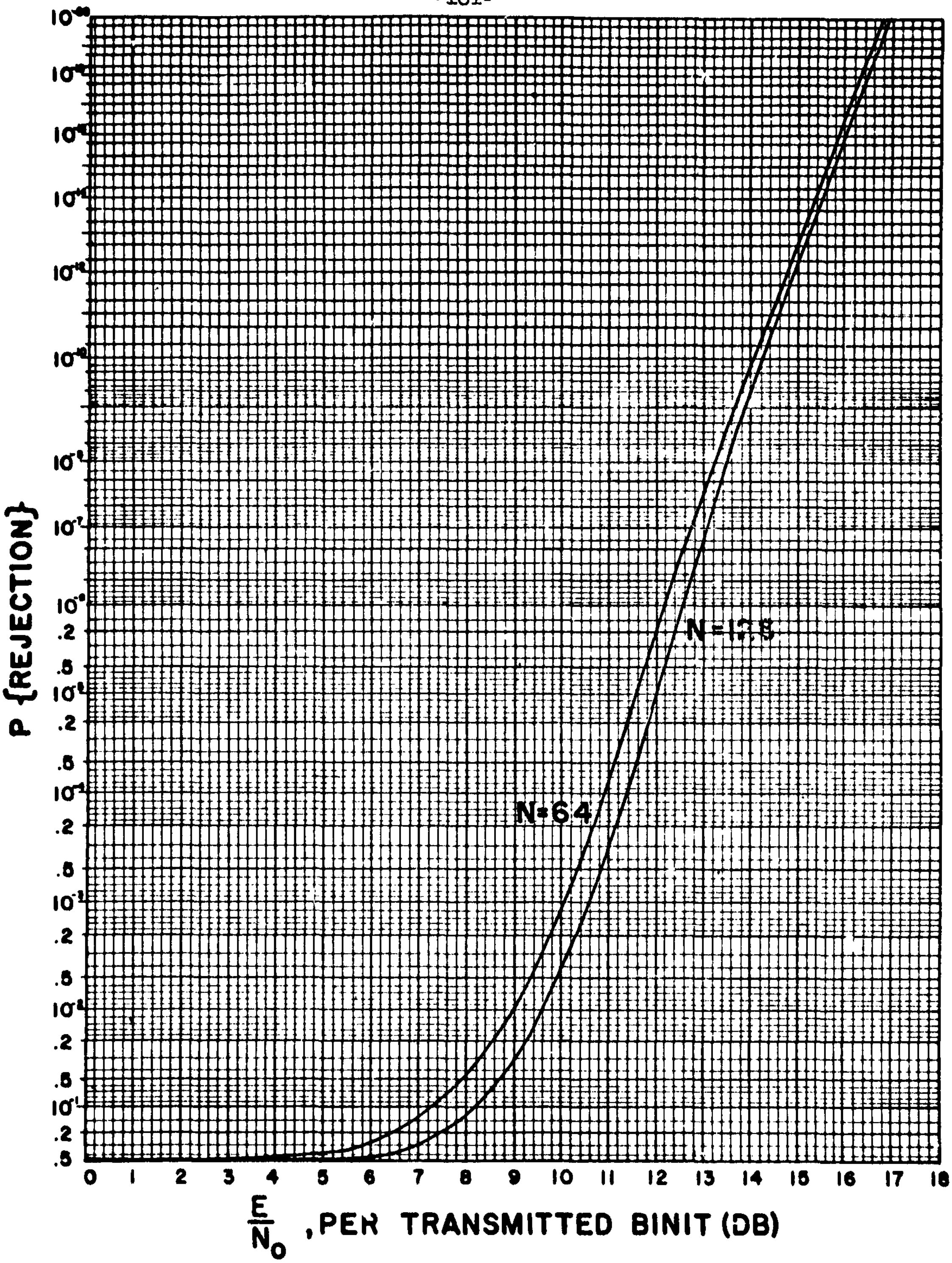
ERROR RATES: B-C (15,5), (15,7) CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.9



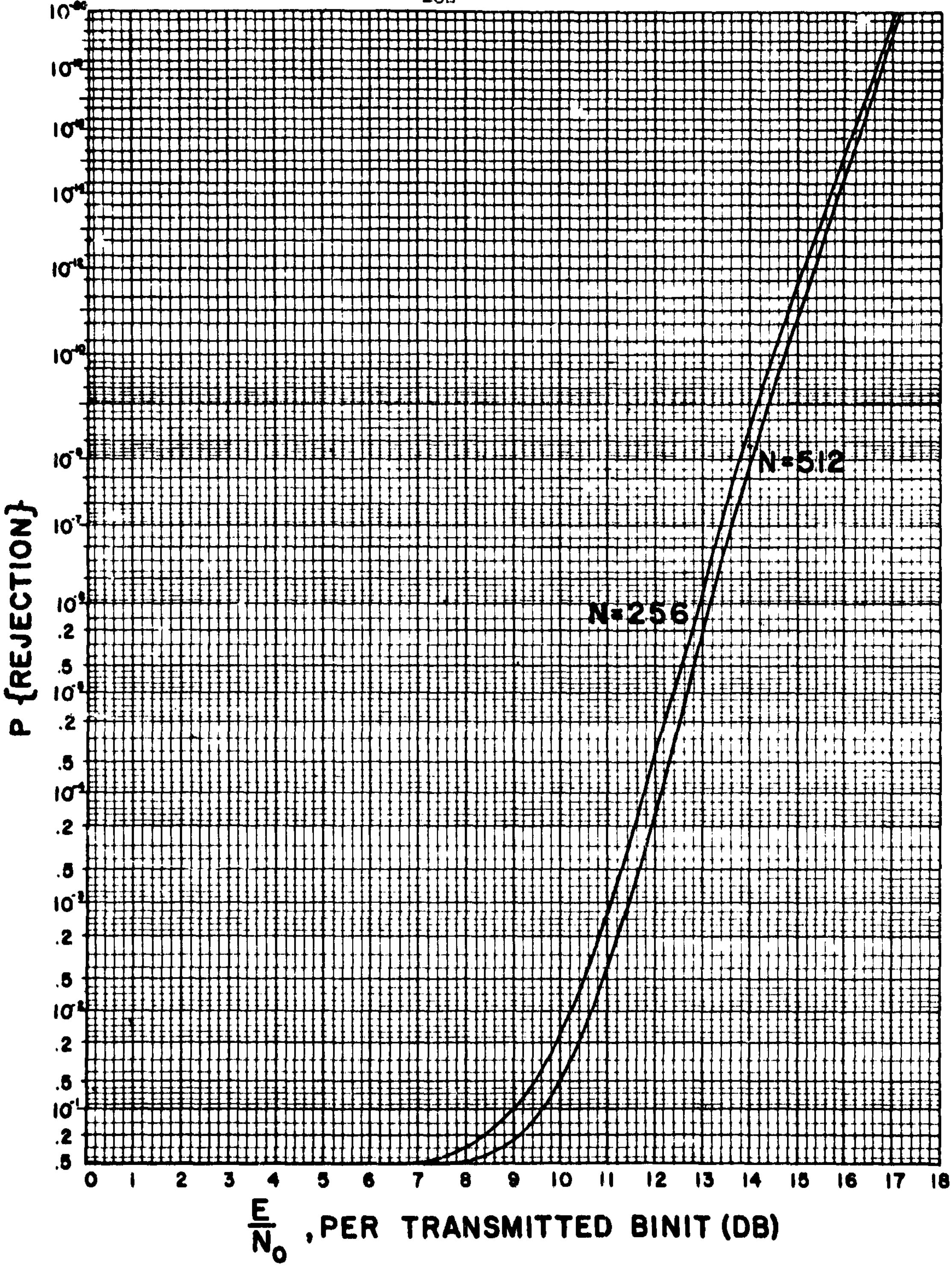
REJECTION RATES: HAMMING SEC/DED CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.10



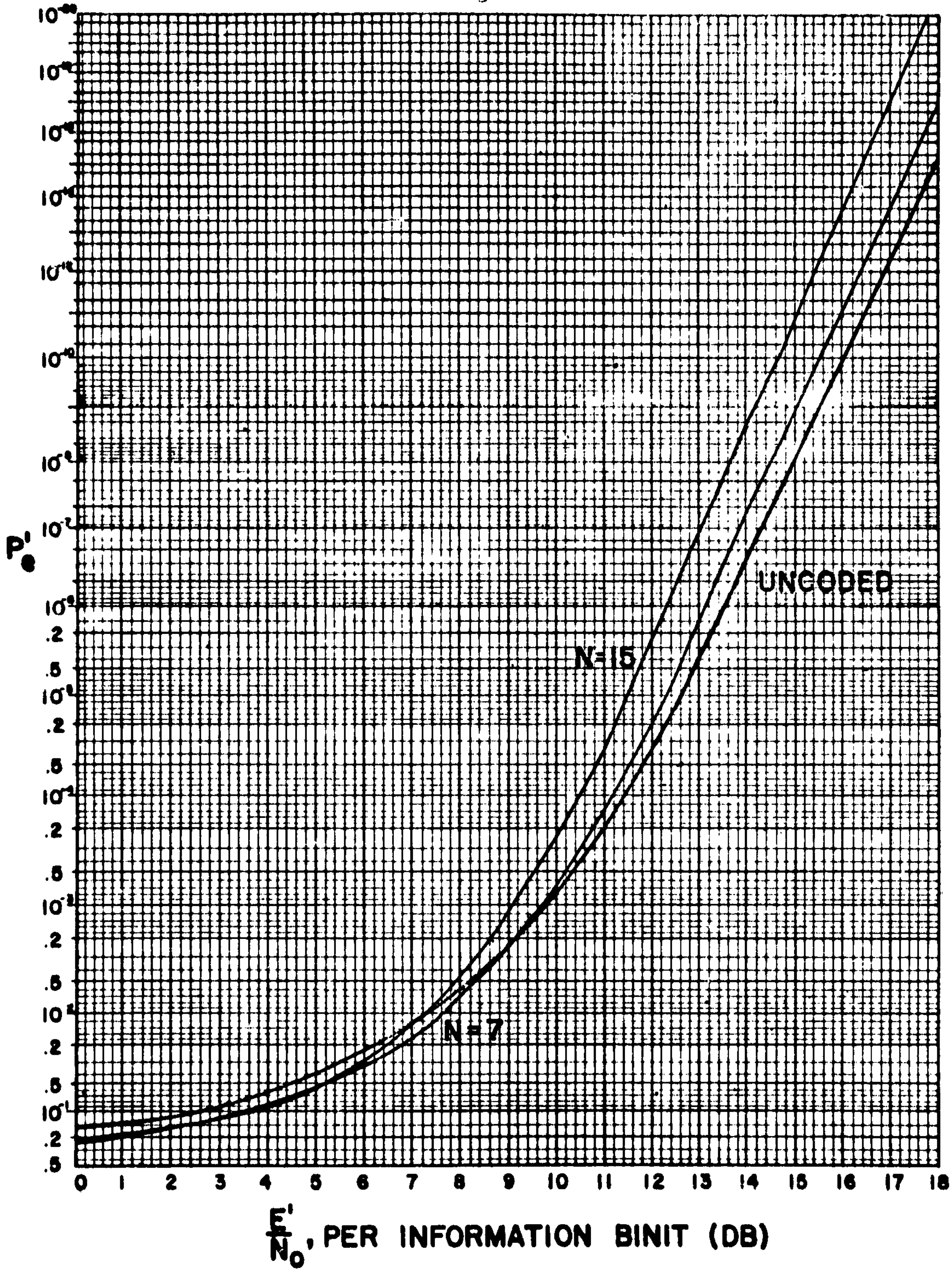
REJECTION RATES: HAMMING SEC/DED CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.11



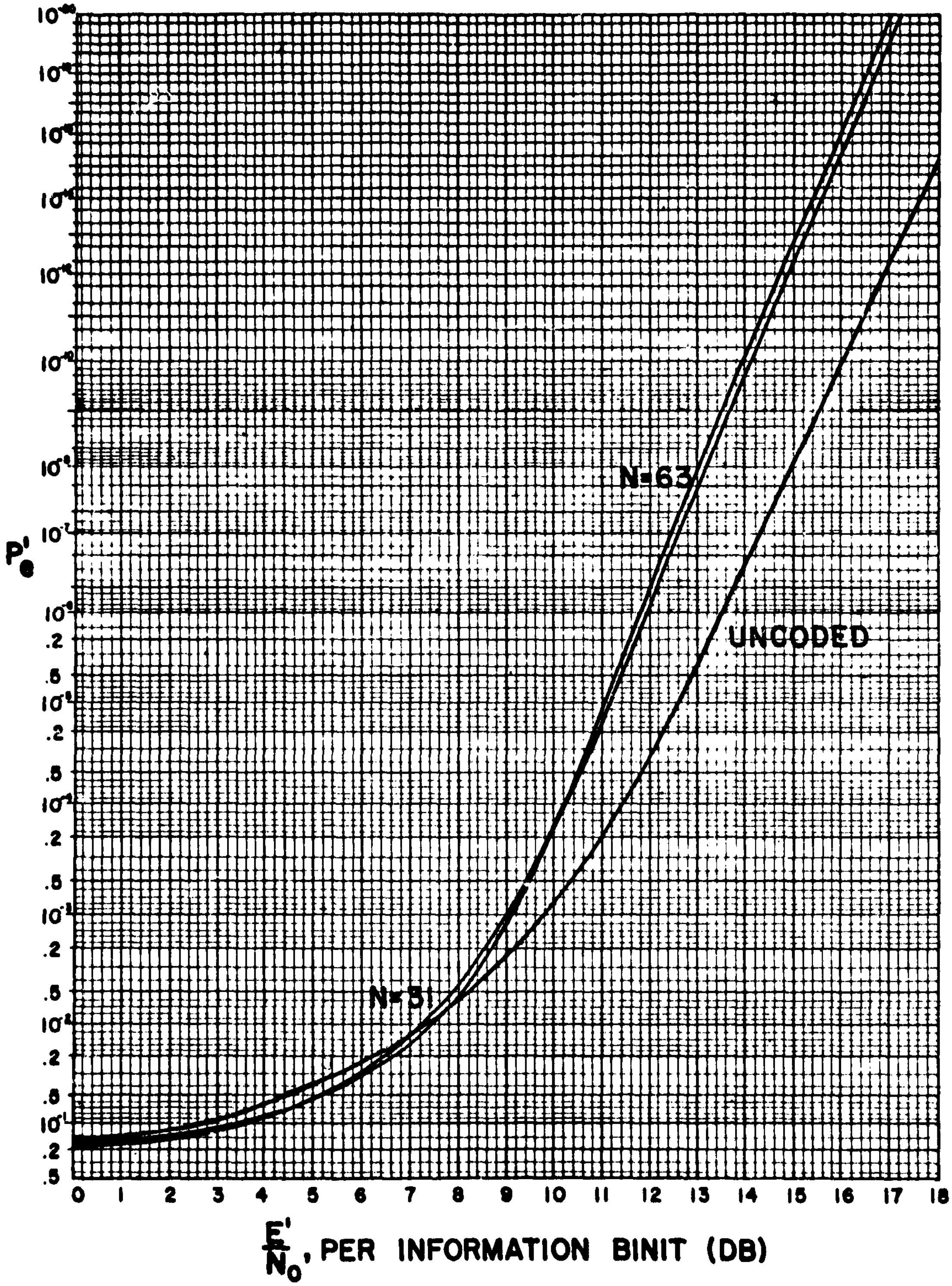
REJECTION RATES: HAMMING SEC/DED CODES - FIXED BANDWIDTH SYSTEM

FIGURE 6.12



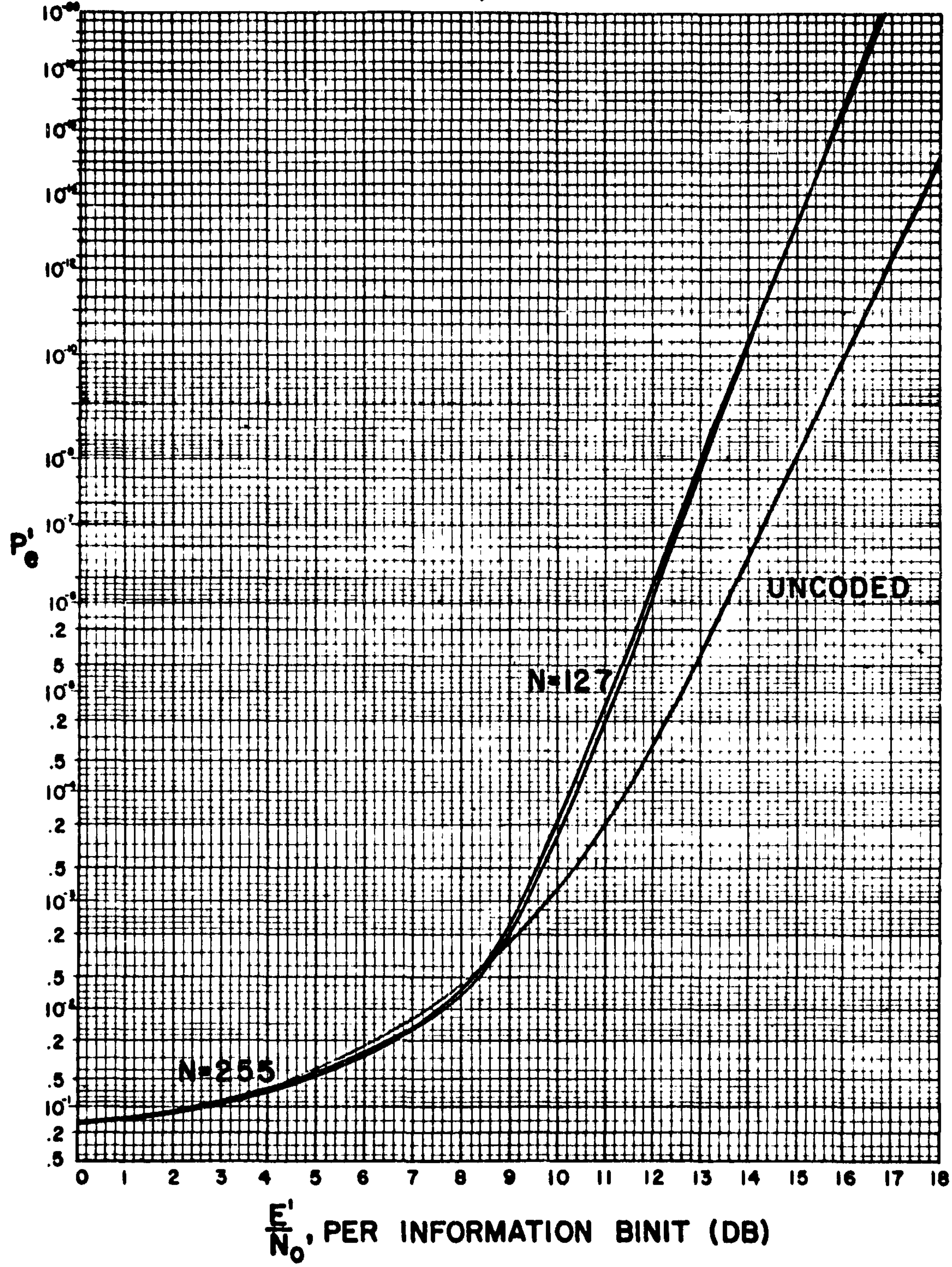
ERROR RATES: HAMMING SEC CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.13



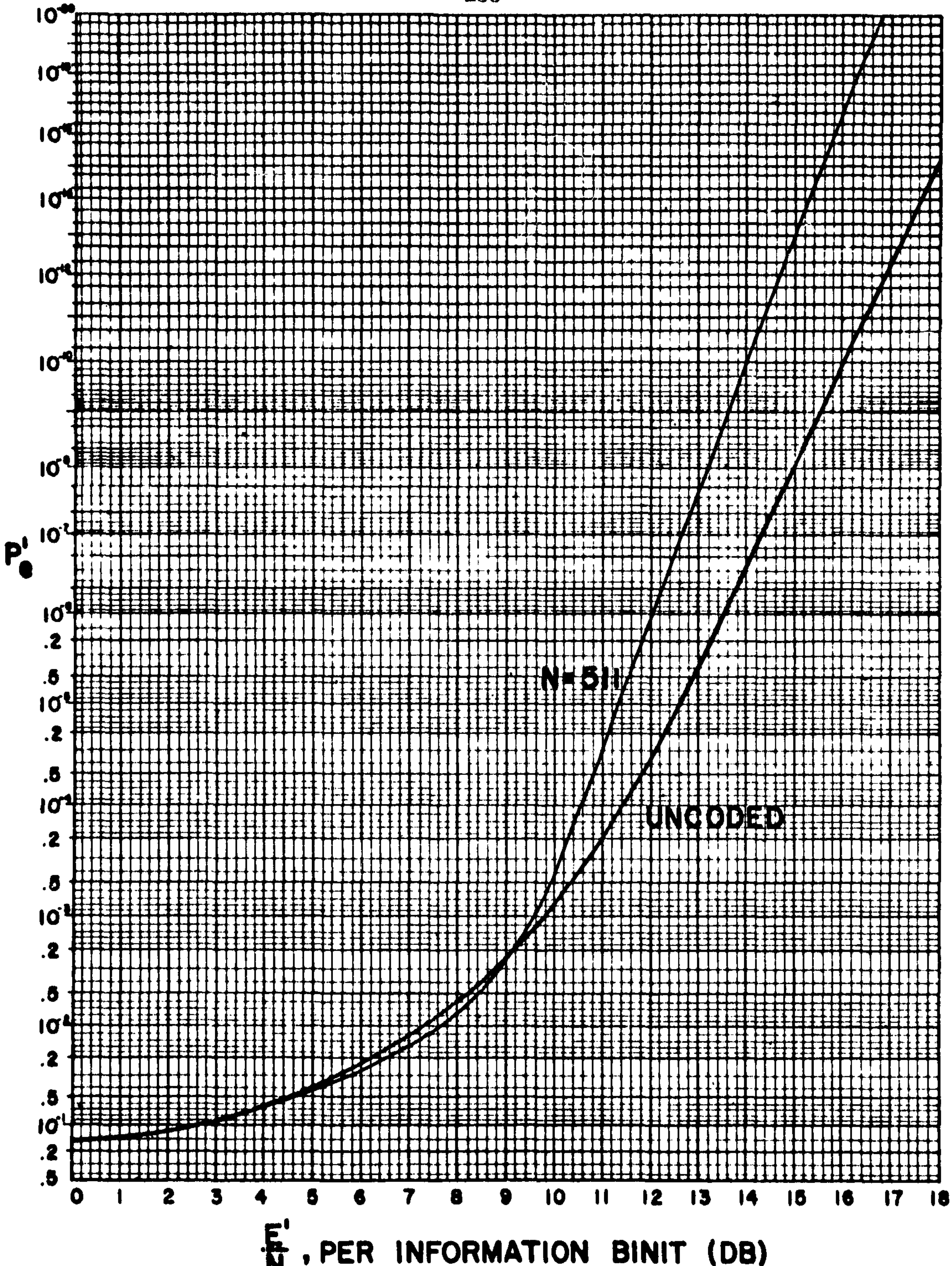
ERROR RATES: HAMMING SEC CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.14



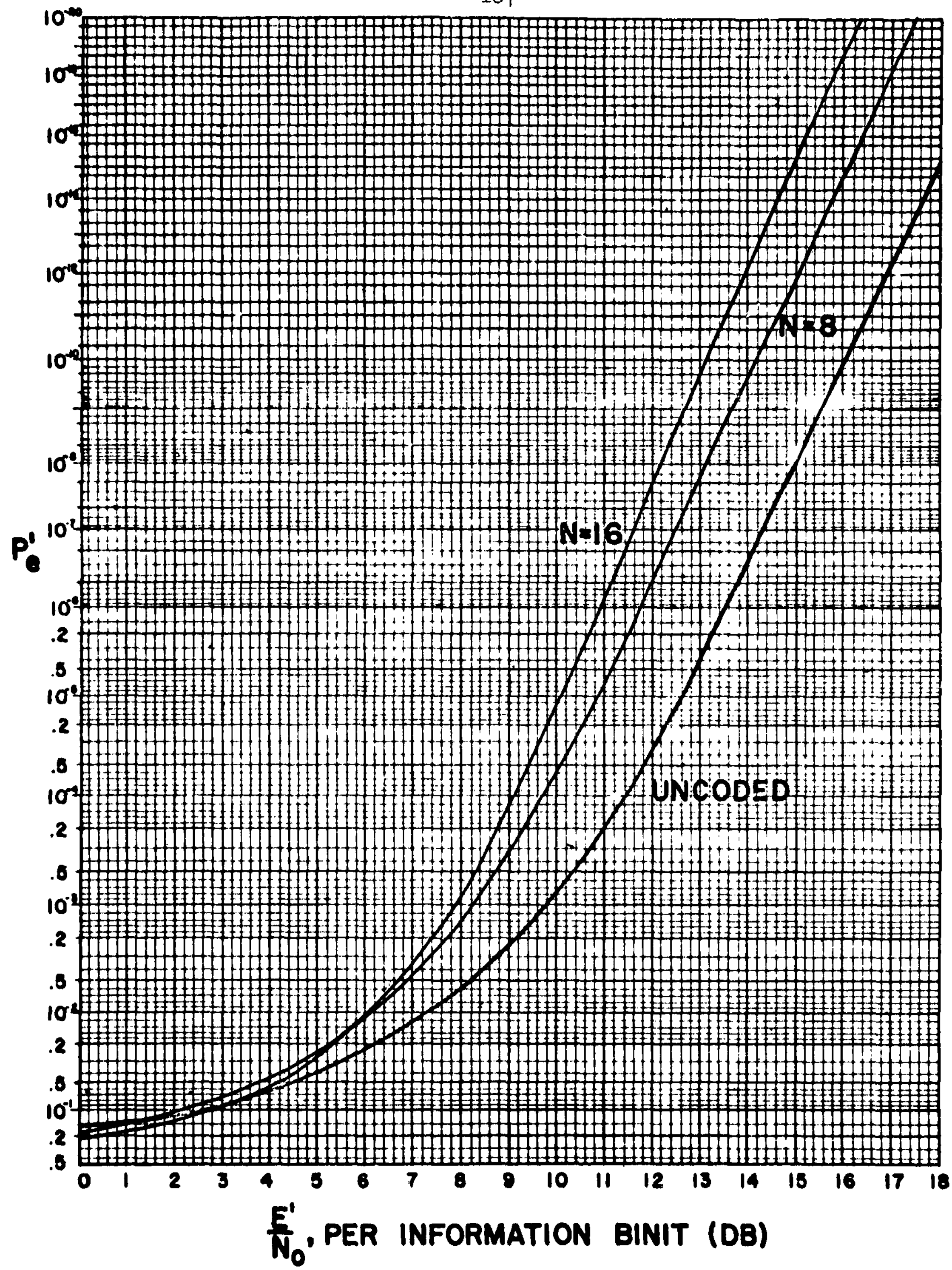
ERROR RATES: HAMMING SEC CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.15



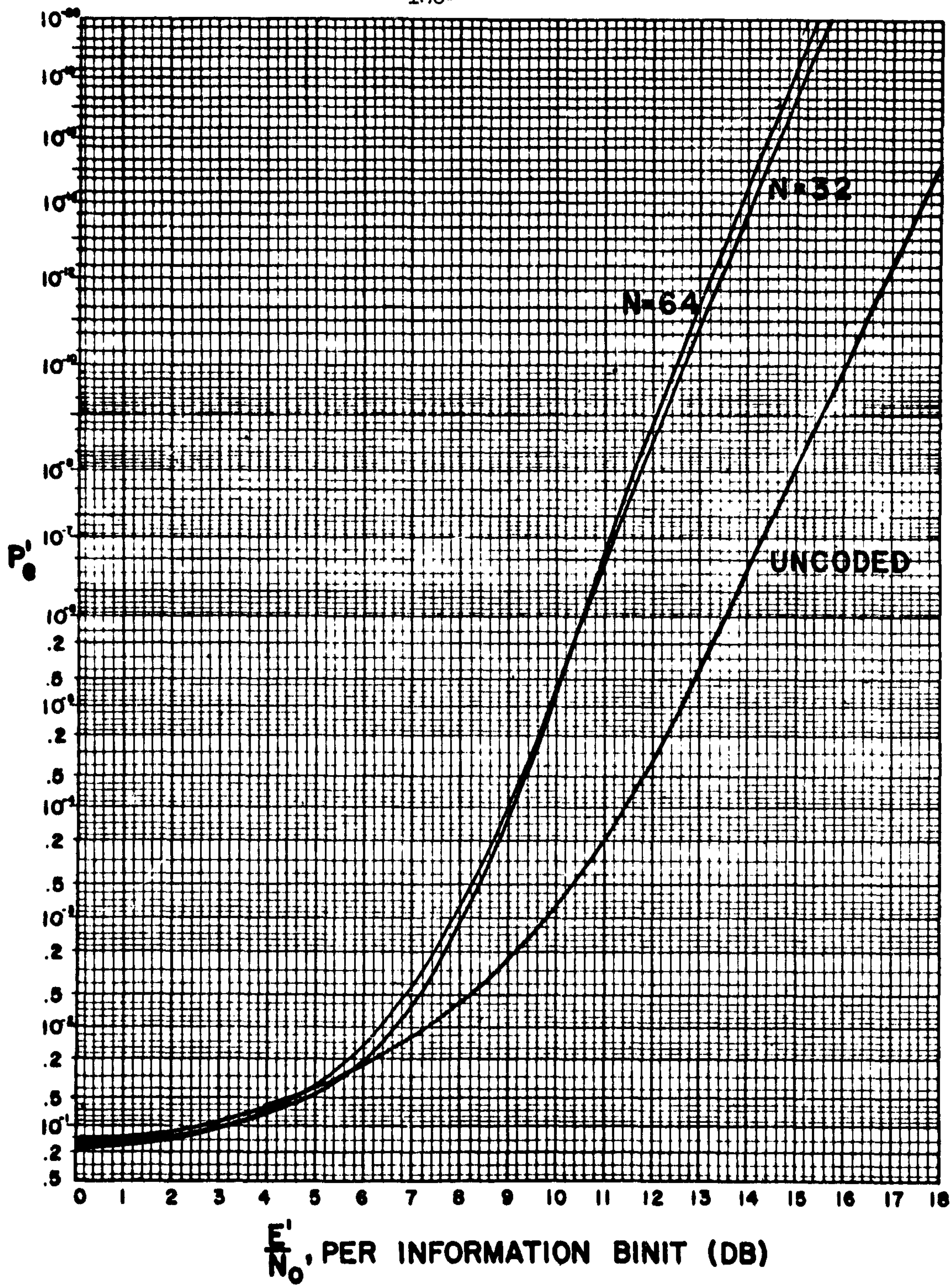
ERROR RATES: HAMMING SEC CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.16



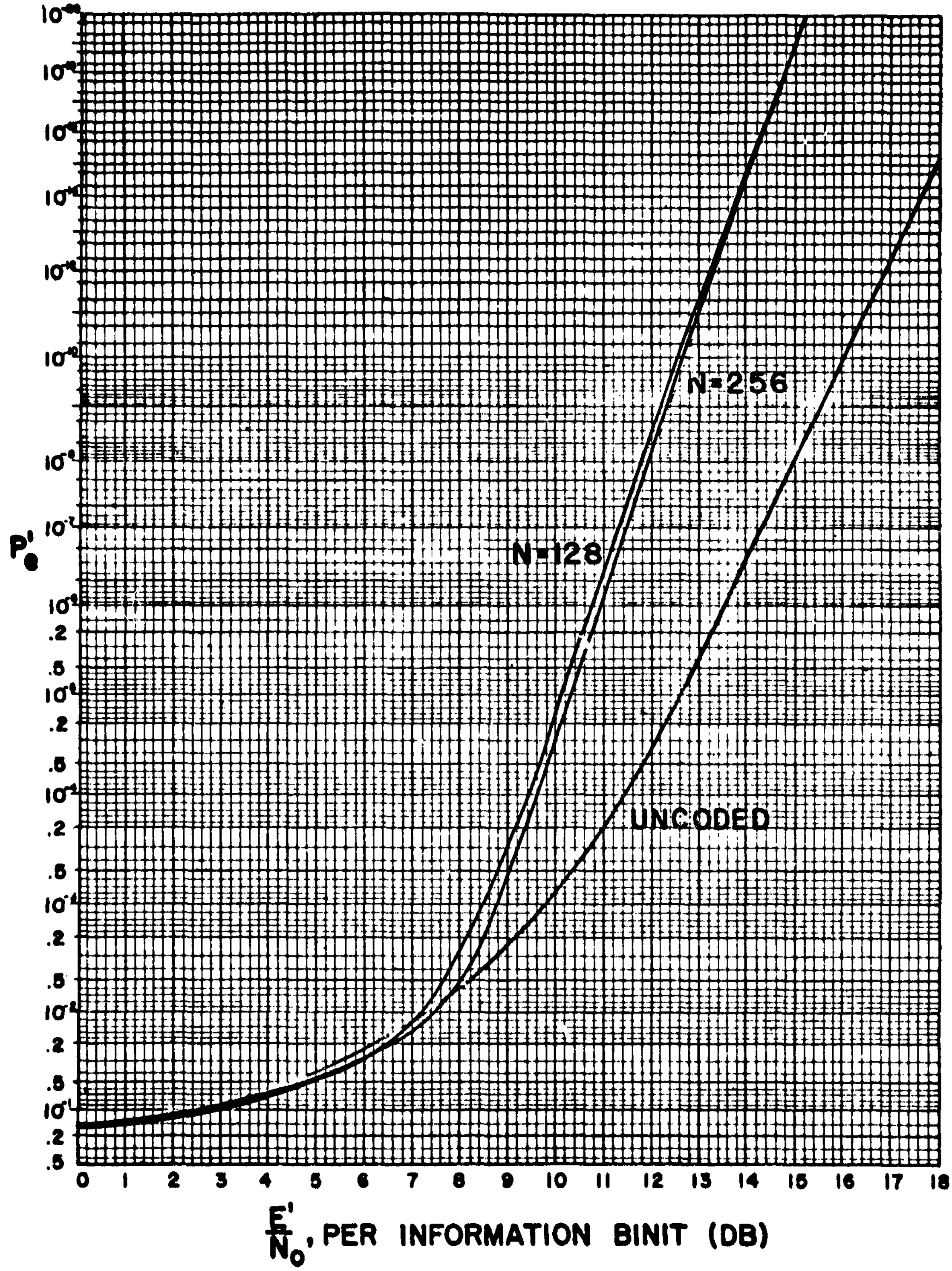
ERROR RATES: HAMMING SEC/DED CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.17



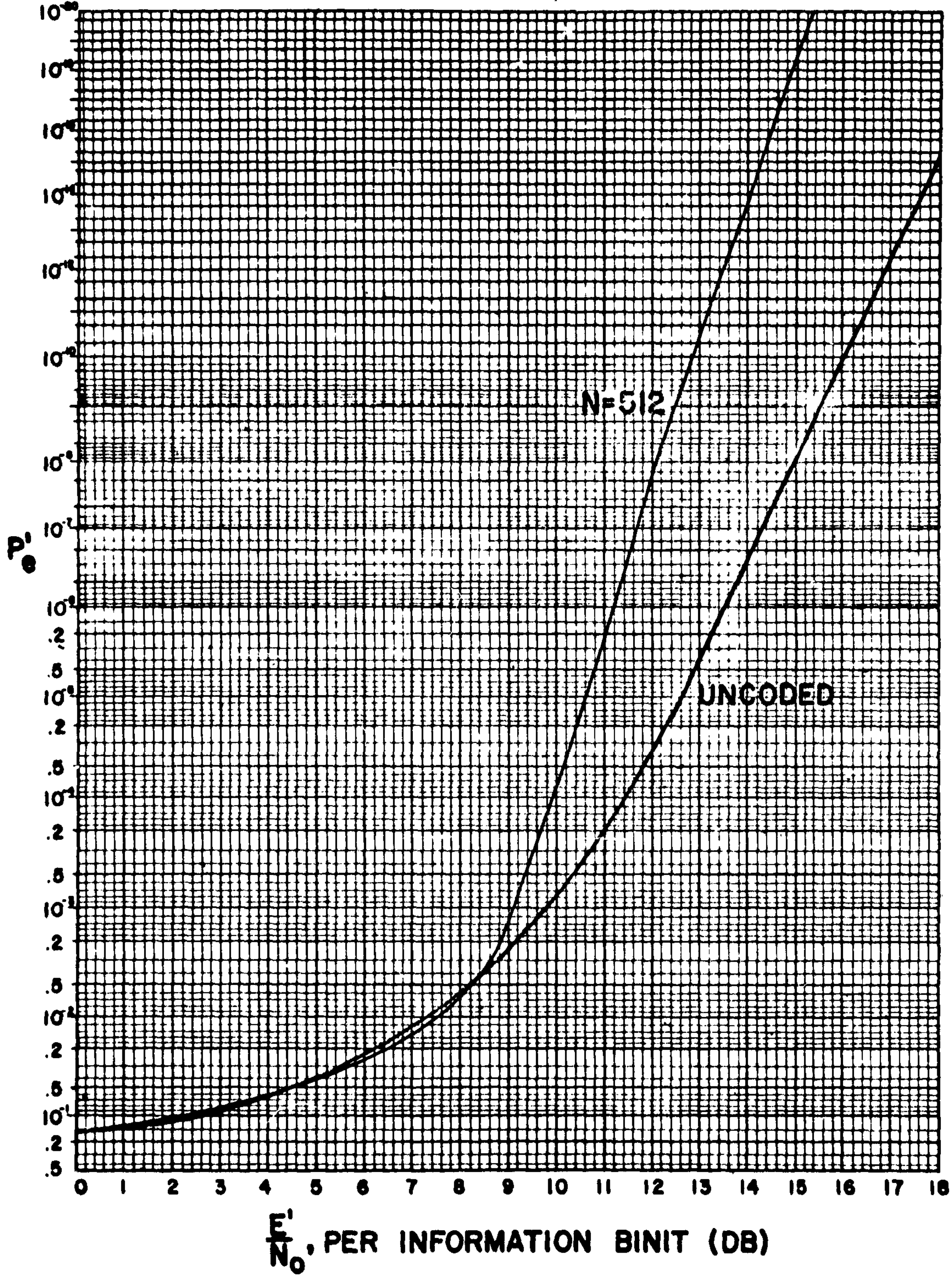
ERROR RATES: HAMMING SEC/DED CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.18



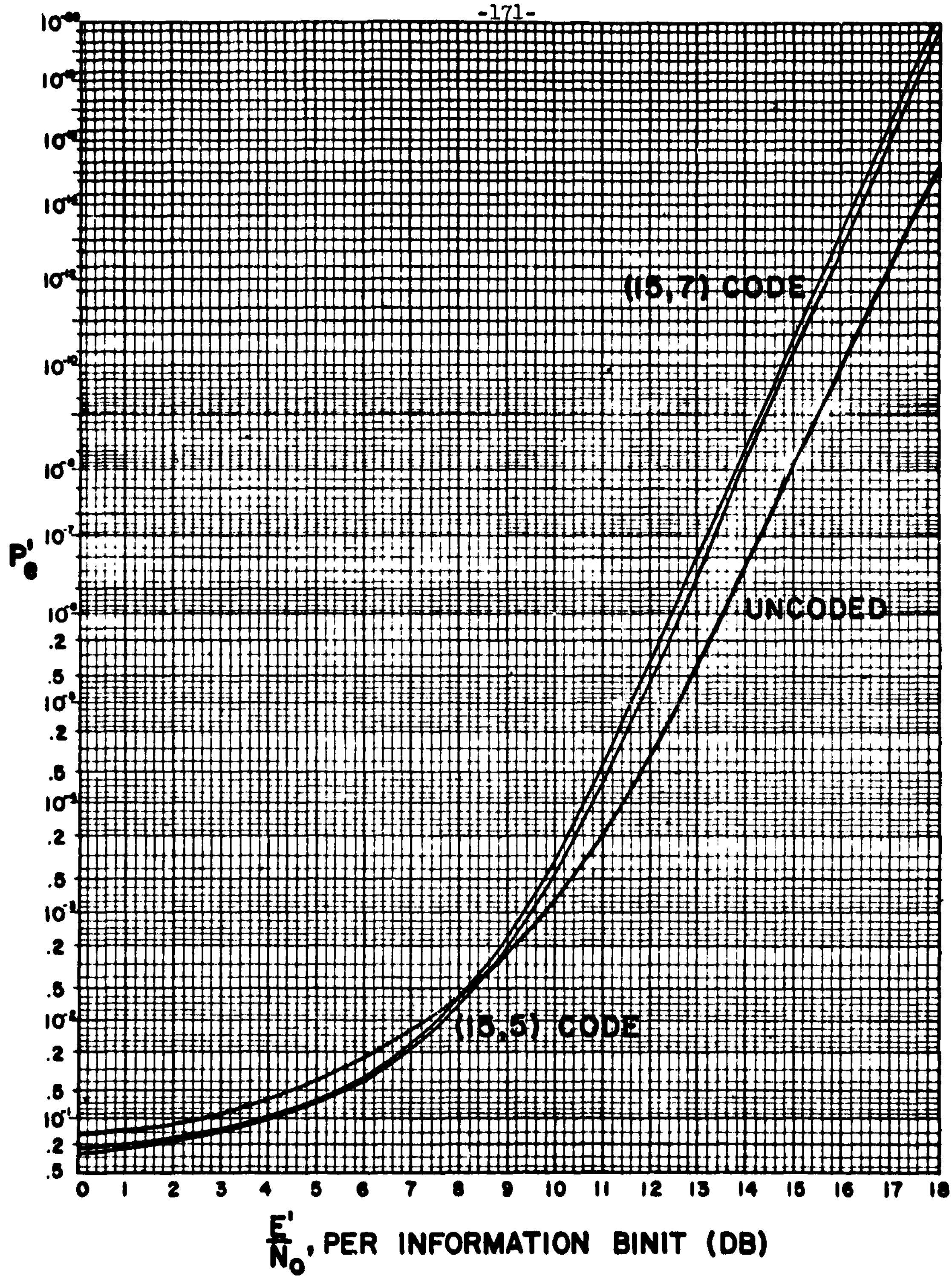
ERROR RATES: HAMMING SEC/DED CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.19



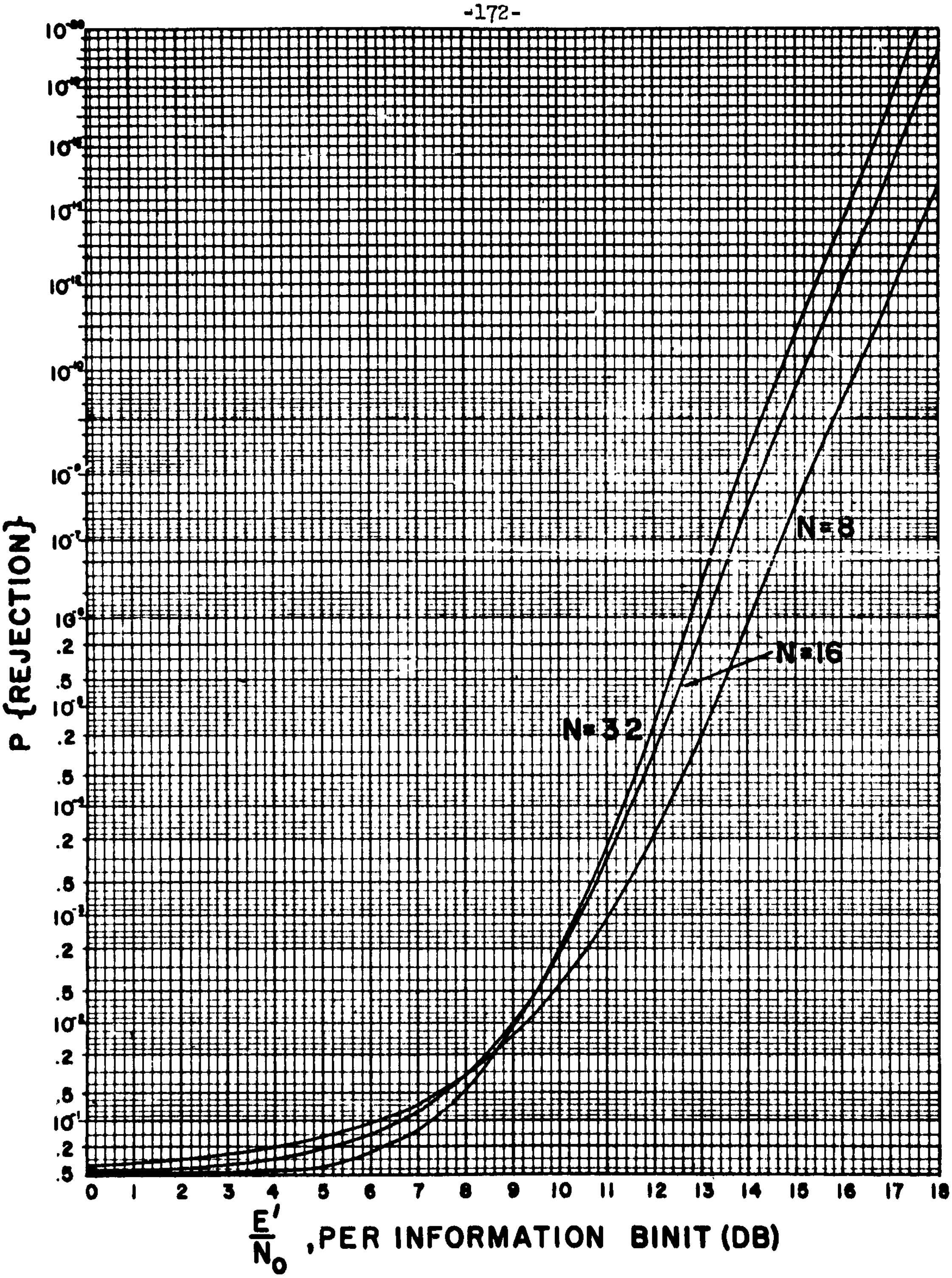
ERROR RATES: HAMMING SEC/DED CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.20



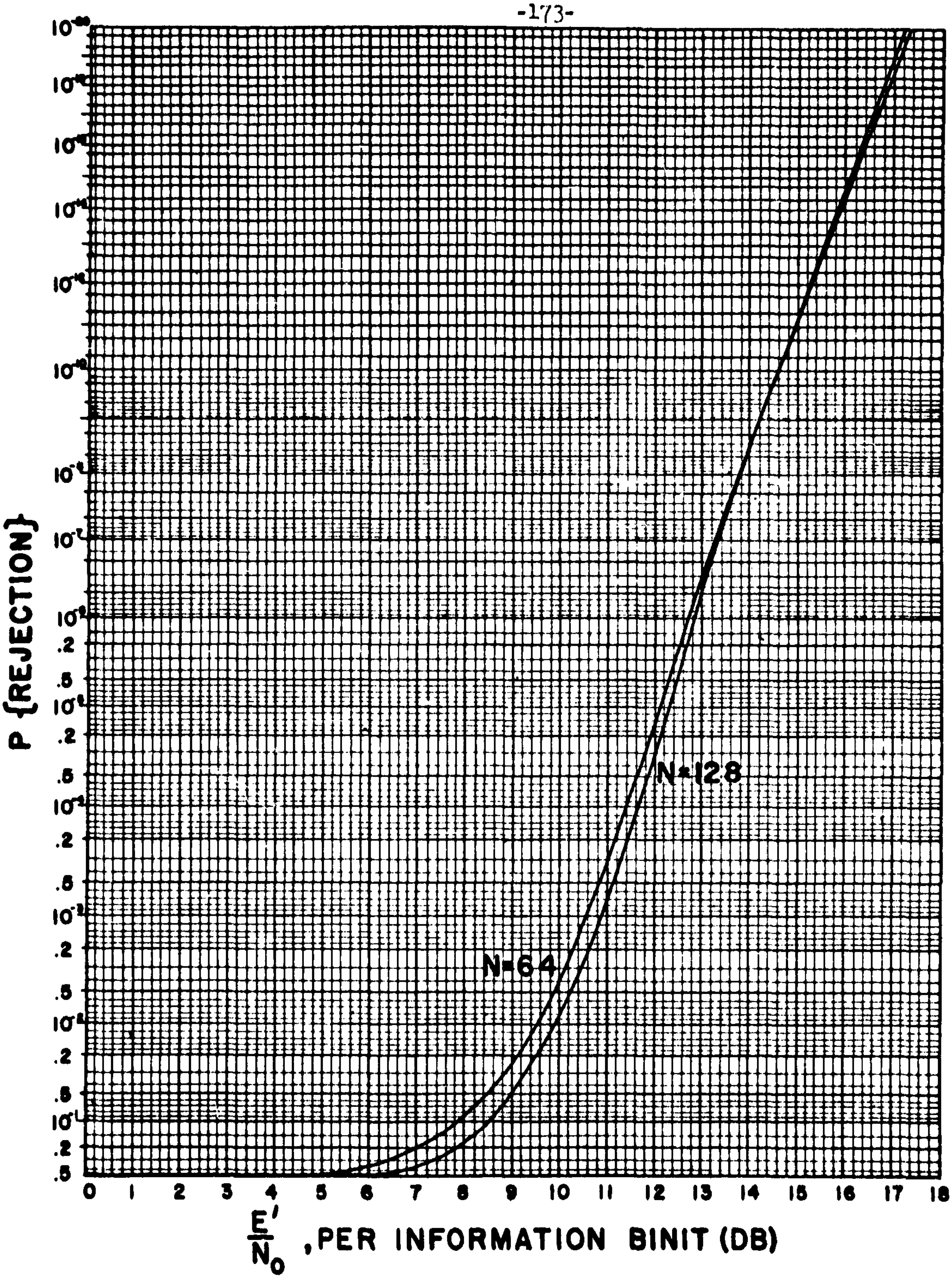
ERROR RATES: B-C (15,5), (15,7) CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.21



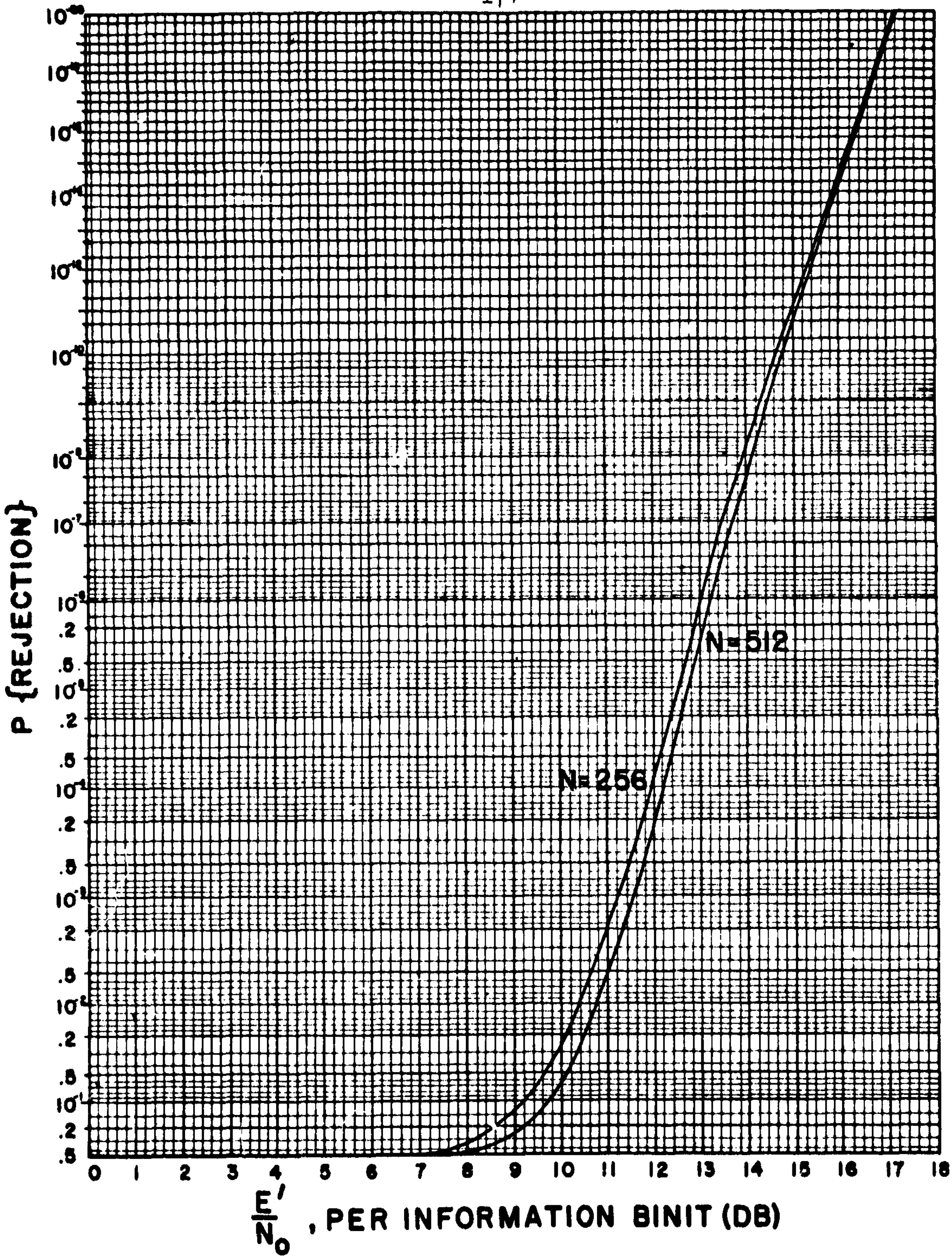
REJECTION RATES: HAMMING SEC/DED CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.22



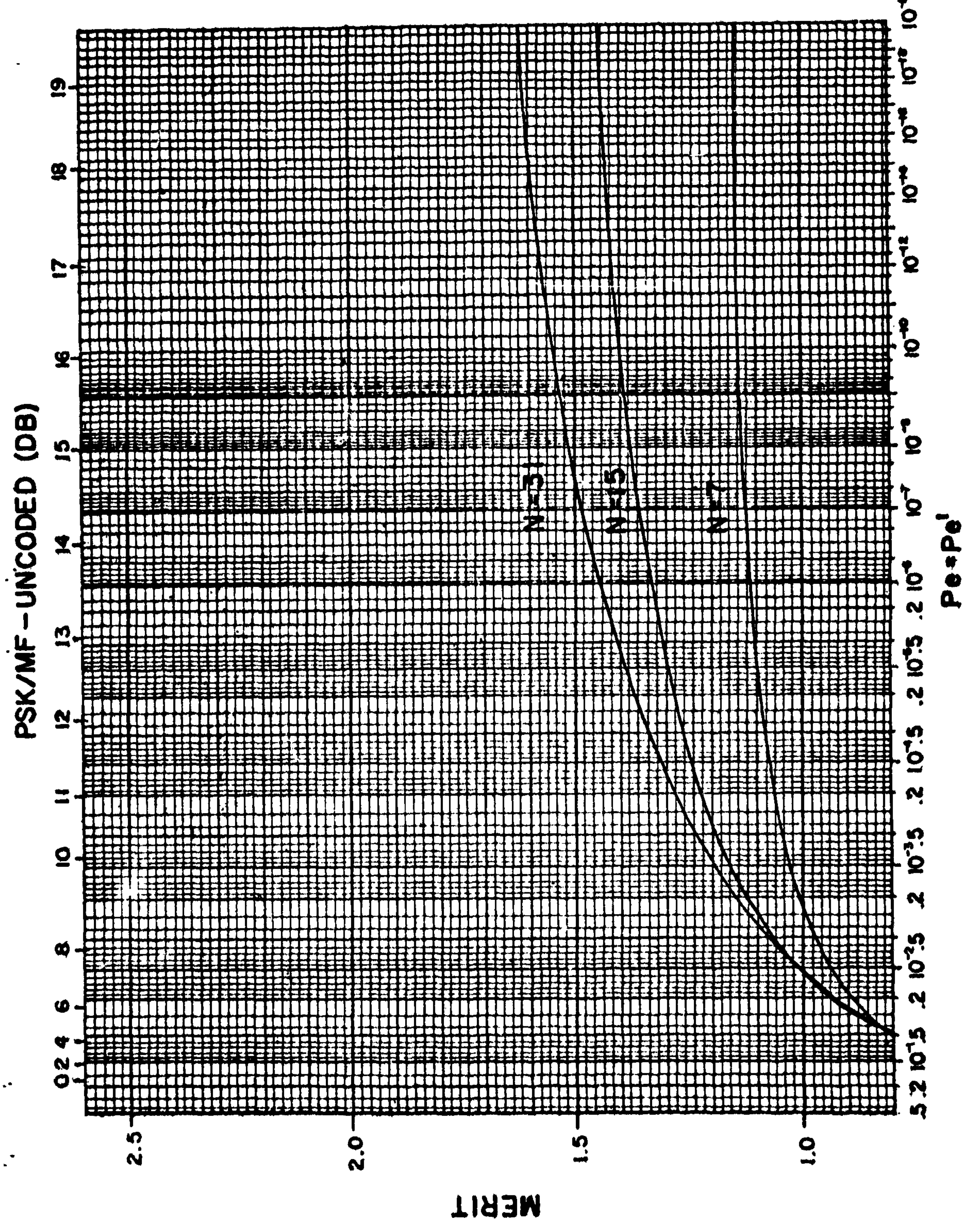
REJECTION RATES: HAMMING SEC/DED CODES - FIXED INFORMATION BINIT RATE

FIGURE 6.23

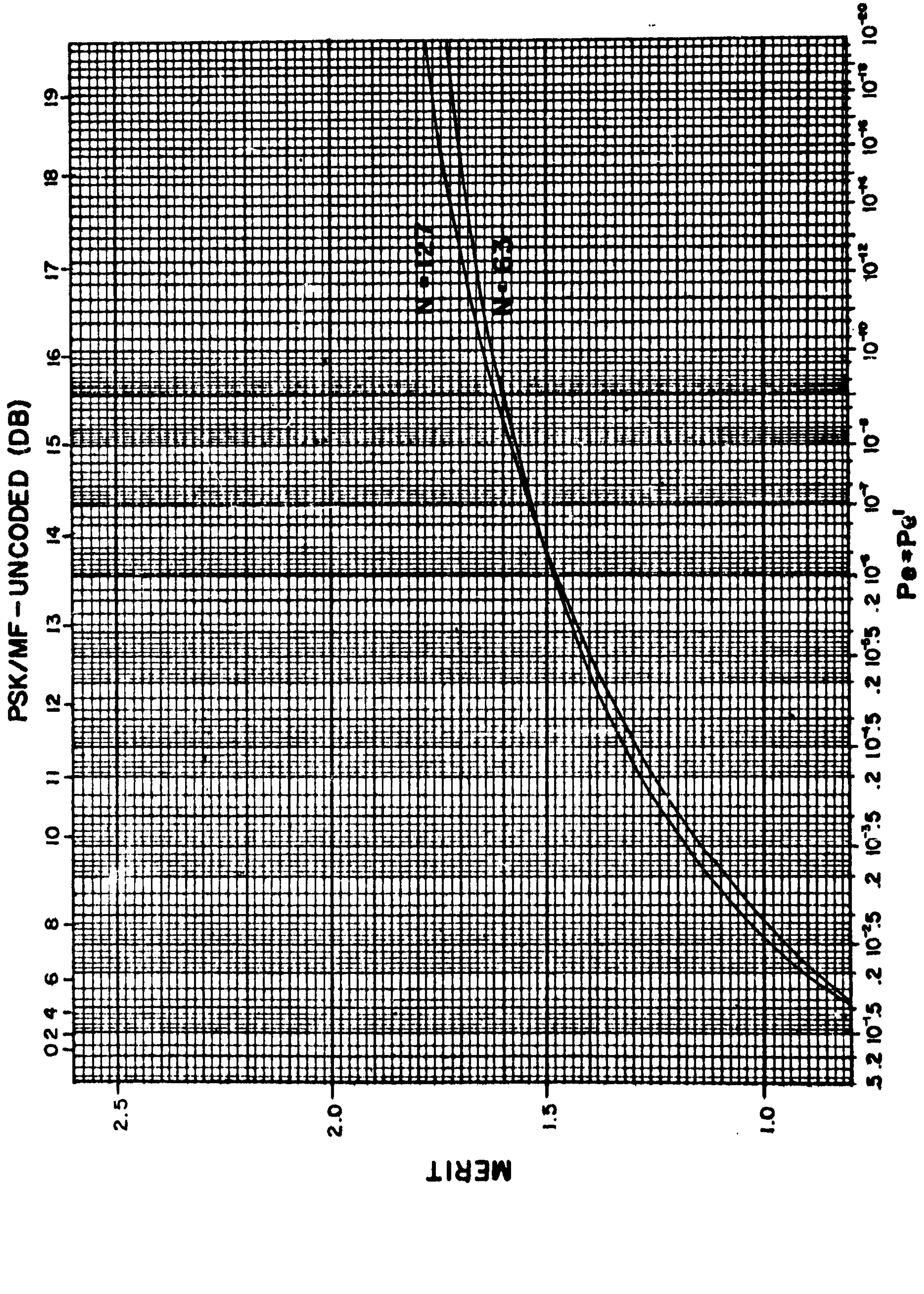


REJECTION RATES: HAMMING SEC/DED CODES - FIXED INFORMATION BINIT RATE

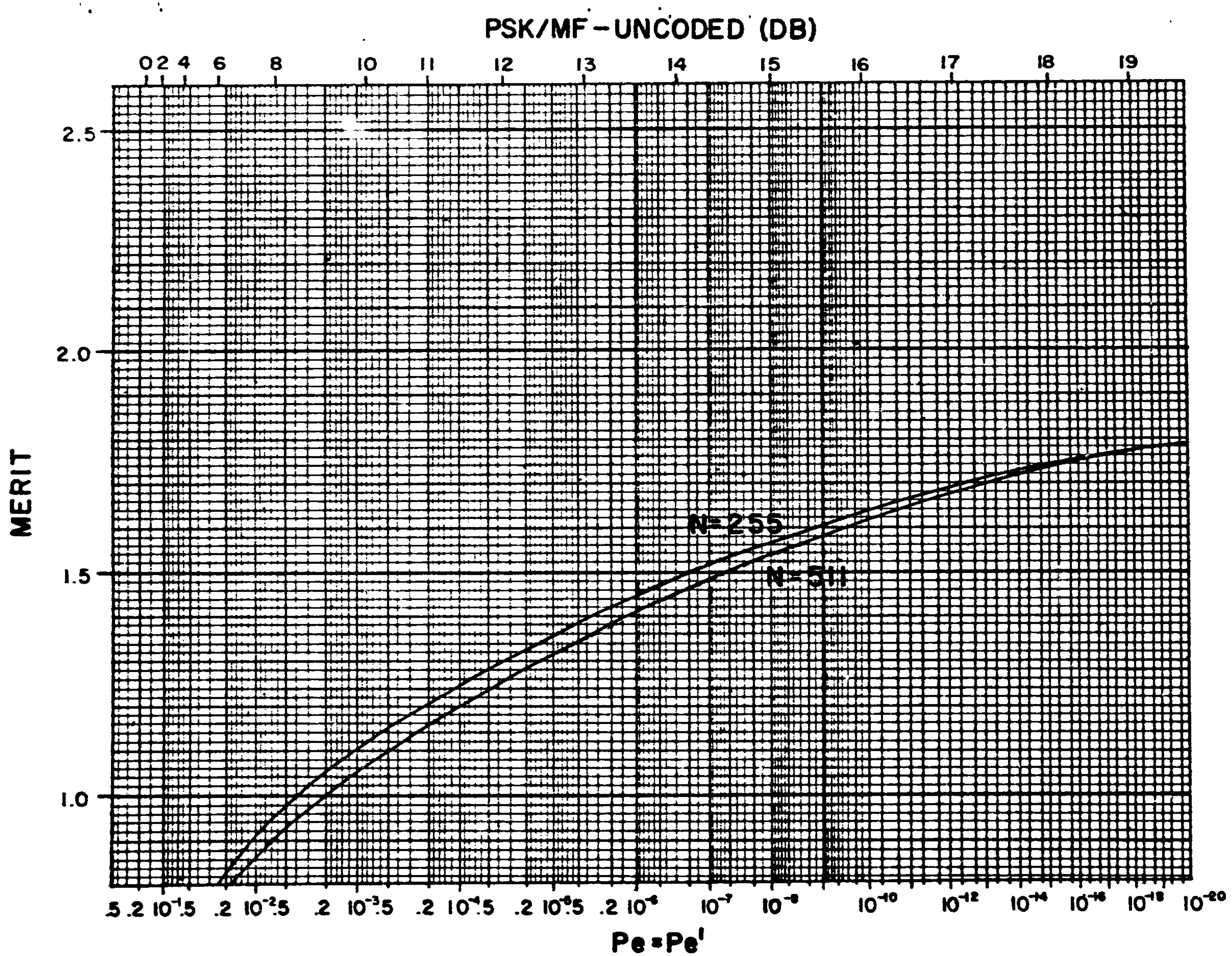
FIGURE 6.24



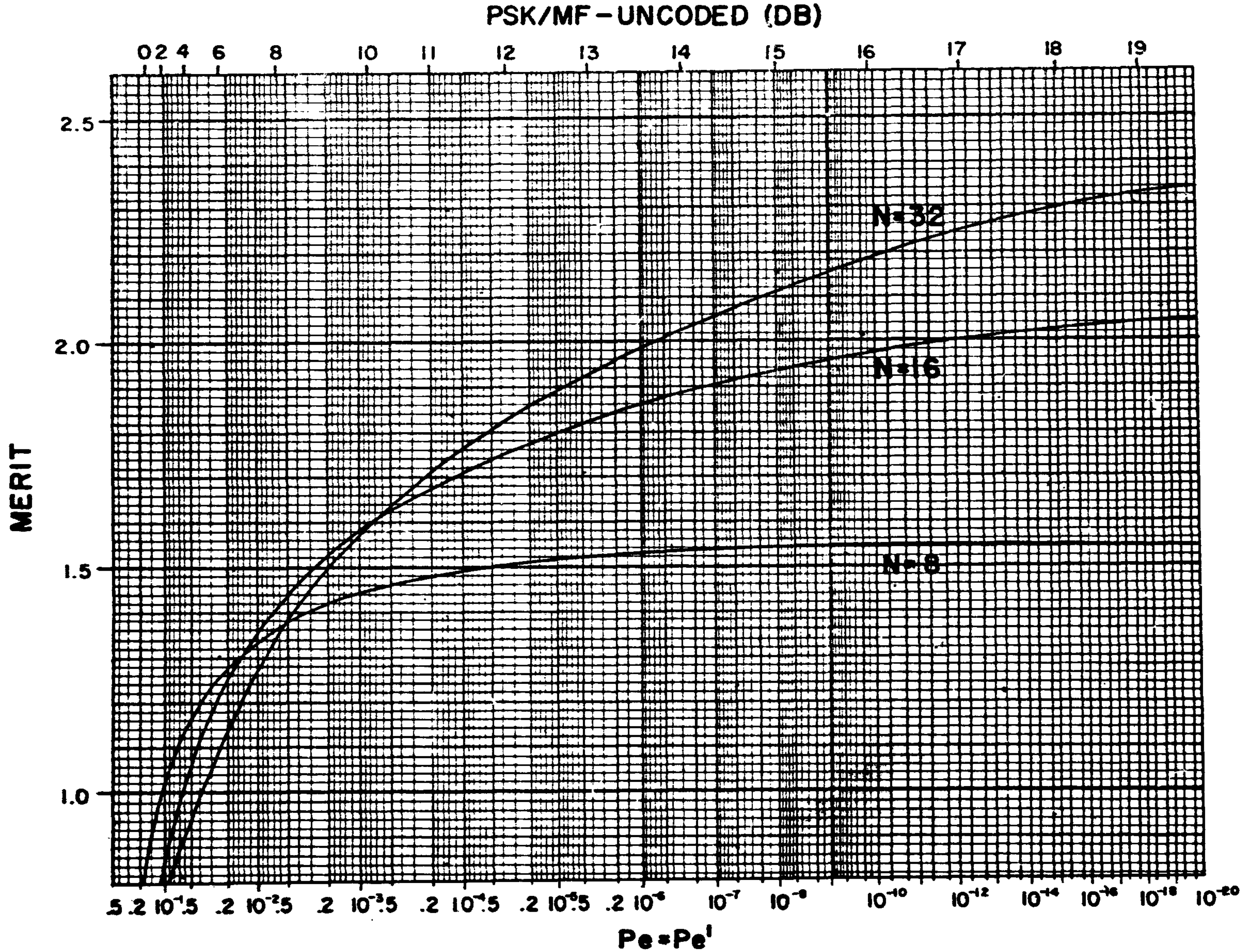
MERIT: HAMMING SEC CODES
FIGURE 6.25



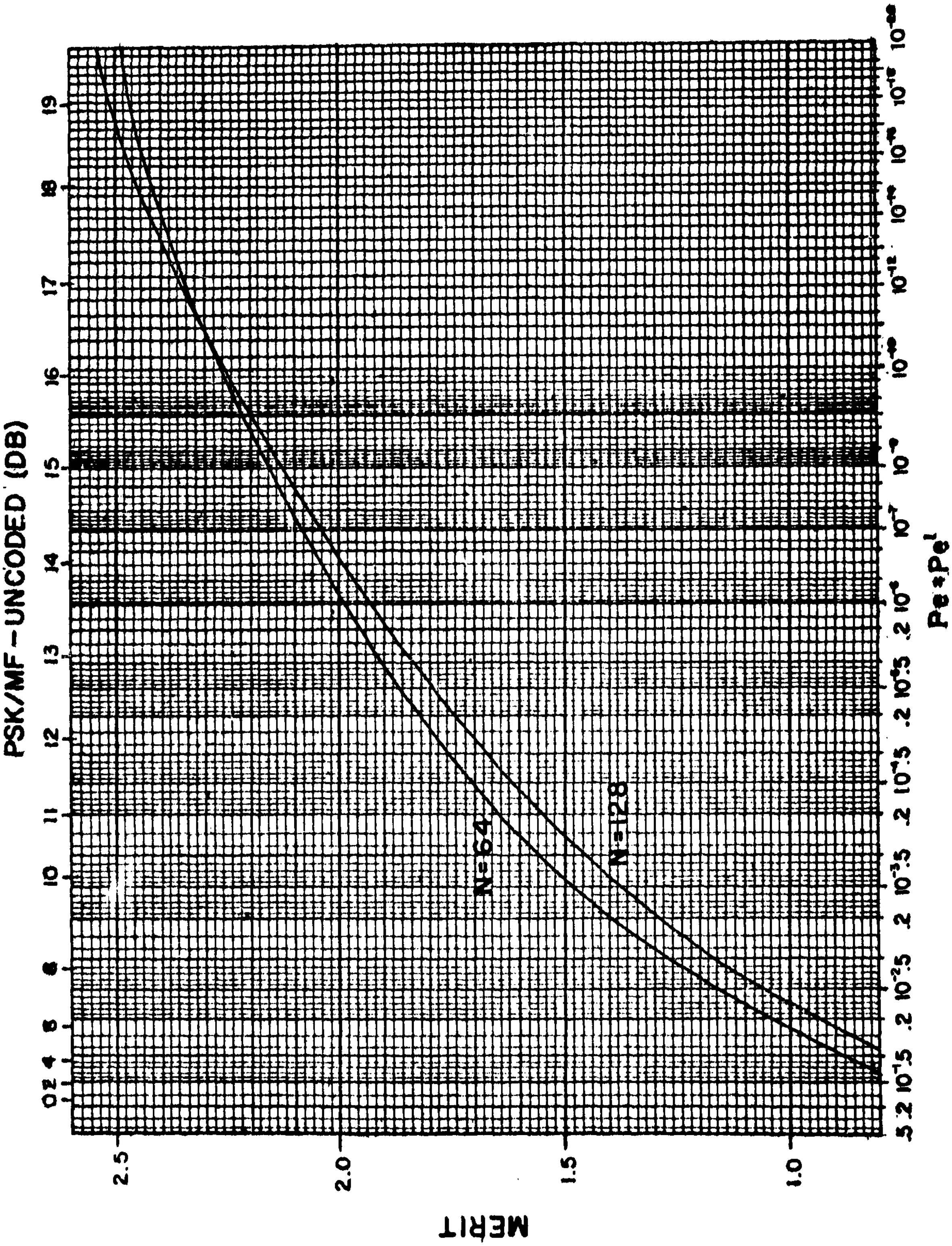
MERIT: HAMMING SEC CODES
FIGURE 6.26



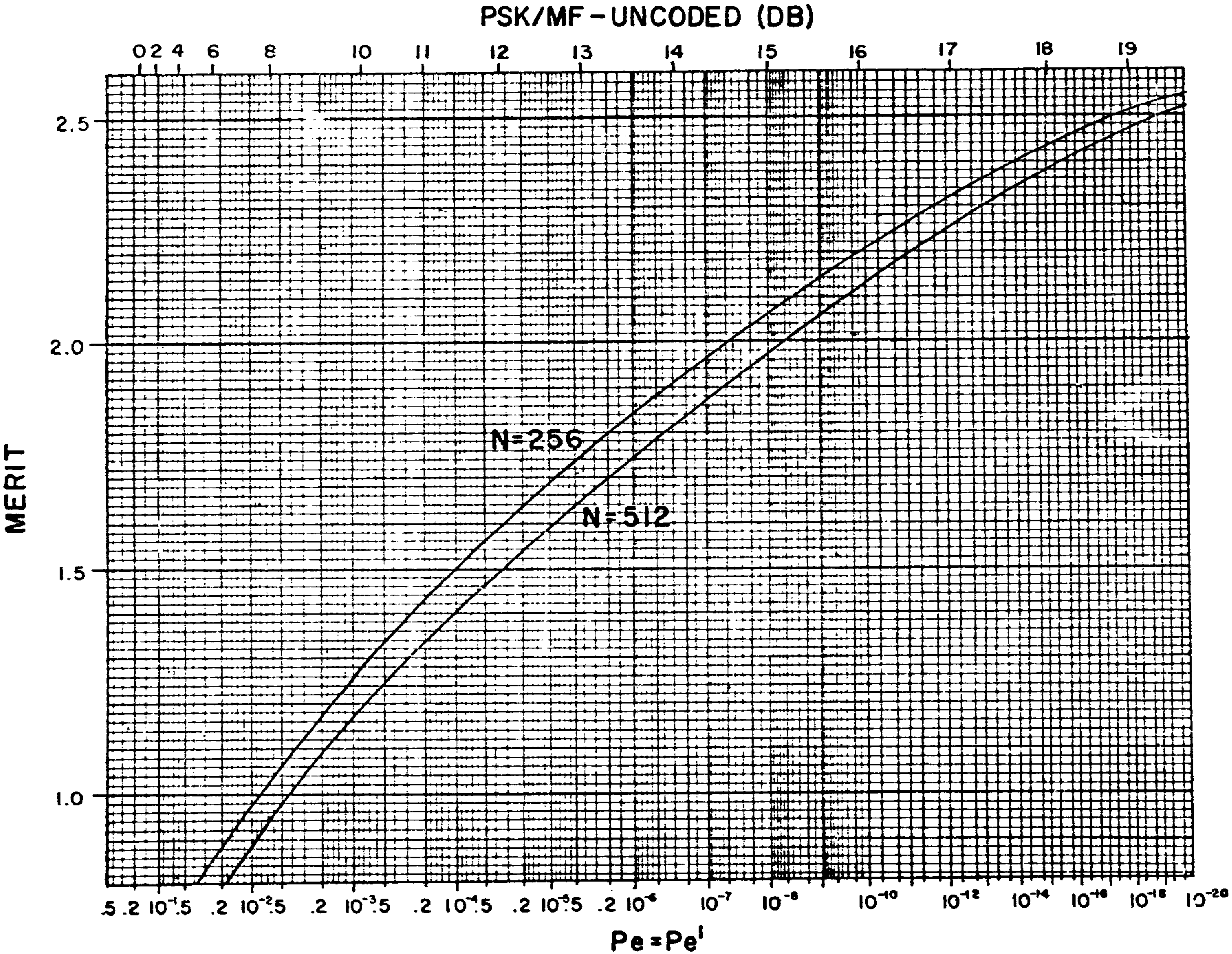
MERIT: HAMMING SEC CODES
FIGURE 6.27



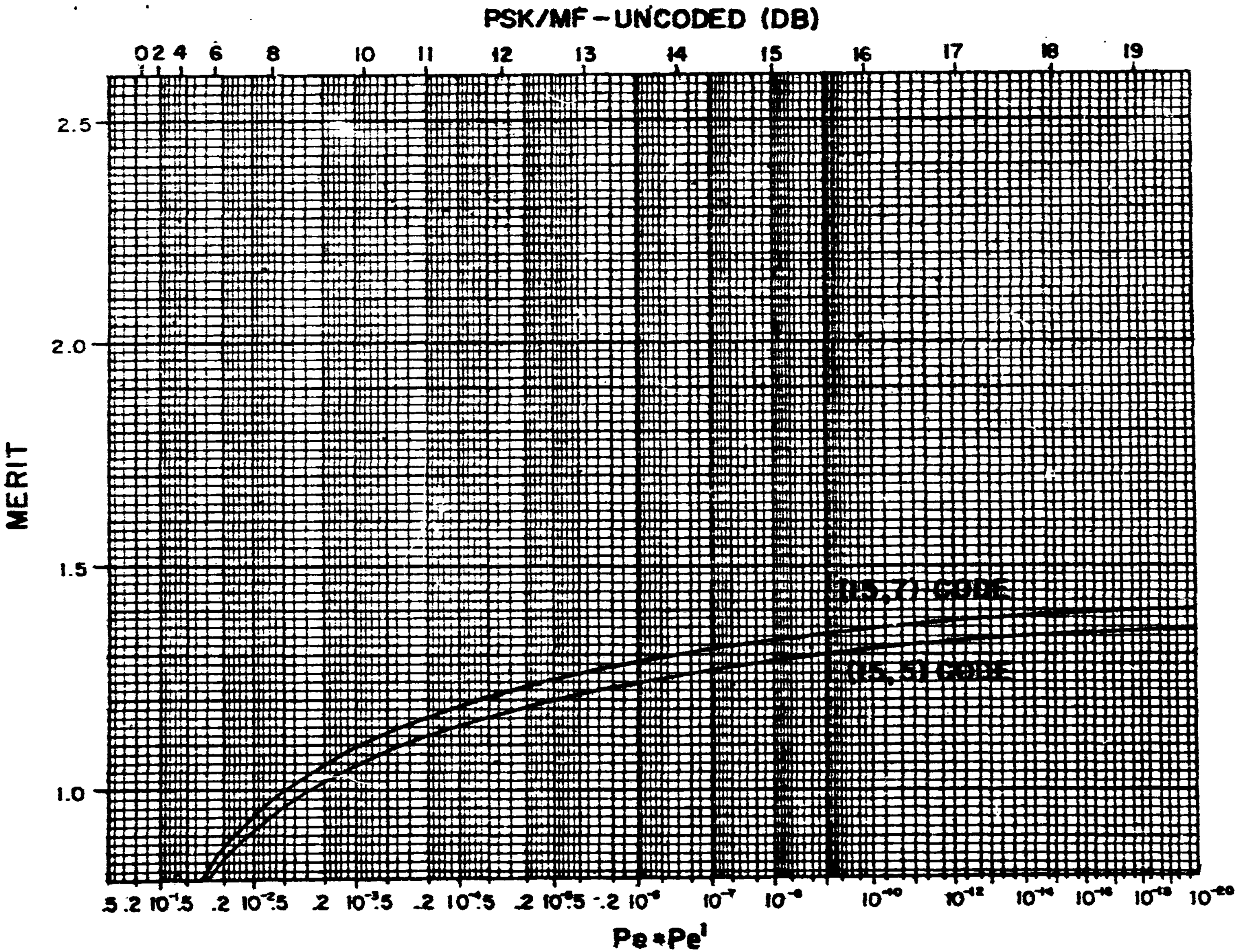
MERIT: HAMMING SEC/DED CODES
FIGURE 6.28



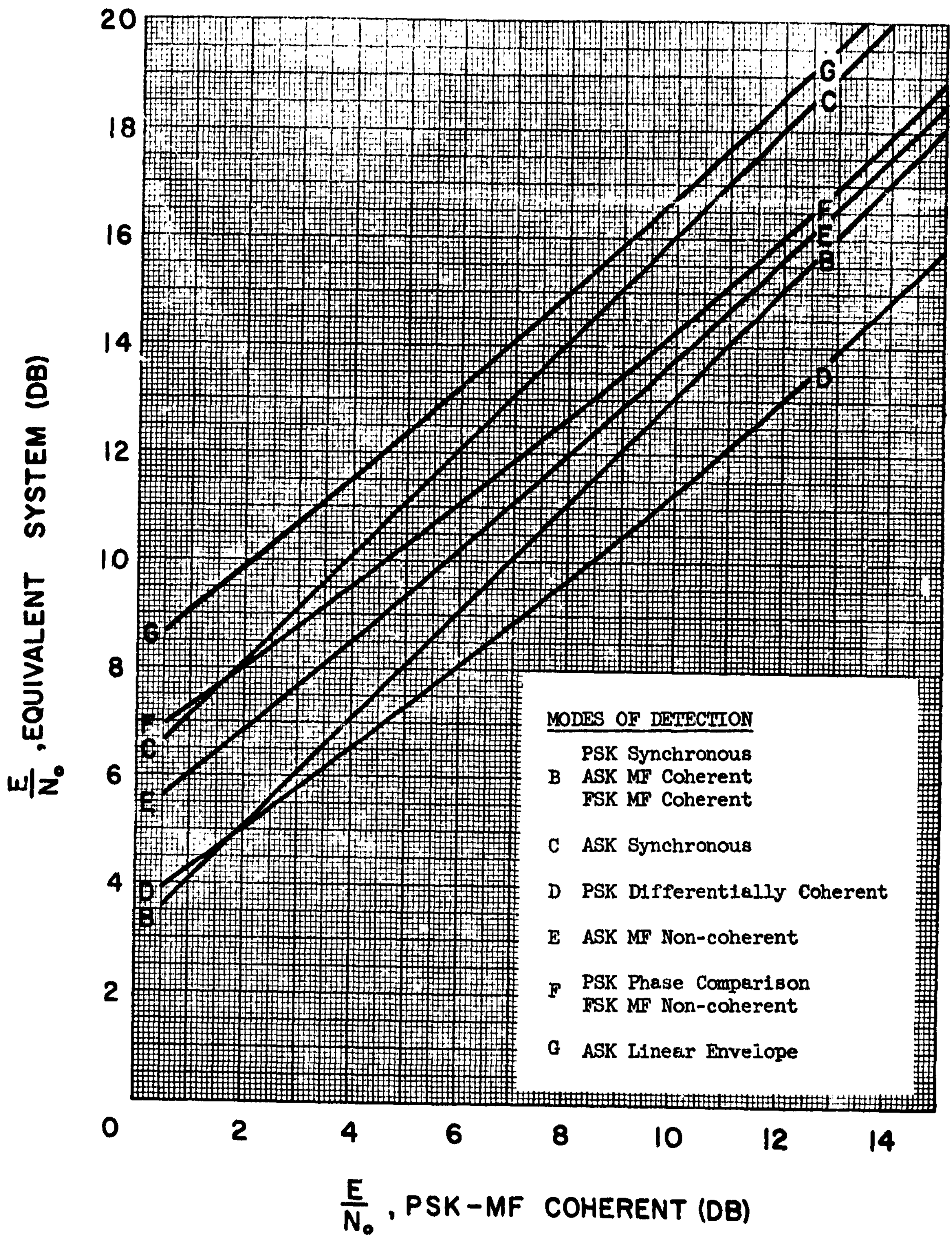
MERIT: HAMMING SEC/DED CODES
FIGURE 6.29



MERIT: HAMMING SEC/DED CODES
FIGURE 6.30



MERIT: B-C (15,5), (15,7) CODES
FIGURE 6.31



MODULATION-DETECTION SYSTEMS: COMPARISON
FIGURE 6.32

CHAPTER VII

SUMMARY AND CONCLUSIONS

7.1 Introduction

This chapter contains a brief summary and recommendations for future work in the areas studied.

7.2 Nonlikelihood Detection Theory

The communication engineer has been frequently faced in the past with the detection of a signal in noise of unknown statistics and will continue to do so in the future, with the increasing importance of outer space travel and of jamming of communications by an enemy. In both of the above cases, it is most difficult to obtain these noise statistics. No detection method presently available will guarantee the required reliability. The theory of nonparametric detection is the only theory applicable and appropriate for these problems. Moreover, nonparametric detection theory is complete in the sense that

(1) It suggests the structure of the detection system which in most cases can be implemented digitally.

(2) It specifies procedures for evaluating the performance of such systems (probability of error, information rate, etc.).

(3) It specifies techniques of system comparison.

The properties of and results concerning nonparametric detectors obtained thus far, were obtained under the severe assumption of independence of the observation samples. This independence is hard if not impossible to be guaranteed since the appropriate sampling times that will result in independent samples are unknown, whenever the probability density and spectral density of the noise are unknown. If one attempts to hopefully

obtain independent samples by sampling at very long intervals, this would decrease the information rate to such an extent as to render the system useless for the transmission of information.

It is, therefore, imperative to establish the validity or not of the results thus far obtained, for the practical case of dependent samples. If the results are valid for dependent samples, this would guarantee a practical and reliable communication system of high information rate even in the presence of noise of unknown statistics. Further extensive research is required to obtain the constant K for the various nonparametric (non-likelihood) detectors and for various actual or simulated channel conditions (tropospheric scatter, ionospheric, line-of-sight transmission, etc.). Knowledge of these constants would permit the quick design of a communication system appropriate to a particular channel condition.

7.3 Optimization of Signaling Waveforms

The investigation reported shows that definite improvements can be achieved in the performance of a communication system by giving suitable consideration to the design of signals. An alternate benefit to be derived from an application of signal design would be the easing of coding requirements while maintaining the same system performance.

Optimum pulse signals have been found for non-overlapping transmission which satisfy the requirement of zero intersymbol interference at the receiver. This optimization has been made for arbitrary signaling rates. The signals obtained in this manner for a given channel can be used for transmission at rates that are sufficiently high to prohibit the use of simple rectangular pulses because these cause excessive smearing of the received waveforms.

It has been shown that for a simple channel model the performance obtained with signals that are optimized for this channel does not

degrade rapidly with changes in the channel characteristics. This is of particular interest in establishing requirements for channel identification measurements.

Finally it has been shown that further performance improvement is possible by permitting successive transmitted waveforms to overlap somewhat.

Only very specific cases have been examined in some detail in this preliminary study. However, the results obtained give some insight into the properties and behavior of signals in digital communications. They also point out the need for much more work in this area. More theory must be developed to treat the problems of signals design, while the results to be obtained are almost certain to greatly benefit the communications art.

Further investigations should specifically be concerned with the following topics:

(1) Continuation of the work presented in this chapter, that is, the optimization of transmission for the system model as described in section 5.2.3.

(2) The application of other performance criteria, such as given in section 5.2.2.2, suitably related to practical system requirements.

(3) Consideration of models for more general types of channels, as listed in section 5.2.2.1, which also includes the problem of specifying appropriate channel models on the basis of specified practical system parameters.

7.4 Performance of Error Correcting Codes

The results contained in Chapter VI cover only the Hamming SEC and SEC/DED codes. Although these codes are the most practical, insofar

as implementation is concerned, there are many other codes whose characteristics warrant further study. Of these, the Bose-Chandhuri t-error correcting codes are particularly important.

Another type of group code is the burst-error correcting code. Unfortunately, standards of comparison of performance for these codes are rather difficult to formulate; and analysis of the causes of burst noise, the duration of the noise, and its effect on binary transmission channels would be a prerequisite to a definitive analysis of a burst-error coded channel. However, one application of burst-error correcting codes for which the basic channel disturbance may be assumed normal is that where the burst code is used in conjunction with the more standard group codes.

Consider a channel using a (15, 11) Hamming SEC code. The information binits at the decoder output are either error free, or contain three or more errors in each group of eleven derived from a single transmitted word - i.e., the errors introduced by the channel, including the coder/decoder, occur in bursts of eleven or less (excluding the possibility of two words both being decoded in error within a short time period). Thus, a further reduction in the error rate may be obtainable by the encoding of the original message by a burst-error correcting code capable of correcting bursts of length eleven or shorter - i.e., the final system would appear as follows:

Of course, the Hamming code may be replaced by a Bose-Chandhuri group code. It is anticipated that such a system would be capable of reducing the error rate to an extremely low value; in fact, such a system would correct the high-order error patterns resulting from a complete channel fade, providing such a fade did not last longer than one Hamming code word.

Another class of codes worthy of study are the sequential codes; these have the advantage of being, in general, easily implemented. No work, so far as can be determined, has yet been done on assessing these codes.

Finally, codes designed around the use of limited feedback channels have not yet been analyzed. Coding for such system is quite different from one way channel coding, and is deserving of separate and complete treatment.

The field of error-correcting code design is so new, and is progressing at such a rate, that very few codes (except for the Hamming codes - in this report) have been analyzed in detail. At this stage, communications system design problems relating to the possible use of error correcting codes cannot, in general, be answered by reference to the existing literature. It is hoped that further research into code performance will fill this void.

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APPENDIX I

EVALUATION OF A CERTAIN INTEGRAL

Consider the integral (Eq. 3-6)

$$C = \max_{\{G_s(f) \geq 0\}} (\ln 2)^{-1} \int_{f_o}^{f_o+W} df \int_0^\infty dx \, \frac{2A}{\sigma^2} e^{-A^2/\sigma^2} \ln \left\{ 1 + \frac{A^2 \, G_s(f)}{G_n(f)} \right\} \tag{I-1}$$

Making a change of variable $y = A^2$ results in

$$C = \max_{\{G_s(f) \geq 0\}} (\ln 2)^{-1} \int_{f_o}^{f_o+W} df \int_0^\infty \frac{e^{-y/\sigma^2}}{\sigma^2} \ln \left\{ 1 + \frac{yG_s(f)}{G_n(f)} \right\} dy \tag{I-2}$$

Integrating with respect to y by parts yields

$$C = \max_{\{G_s(f) \geq 0\}} (\ln 2)^{-1} \int_{f_o}^{f_o+W} df \, \frac{G_s(f)}{G_n(f)} \int_0^\infty \frac{e^{-y/\sigma^2}}{\frac{1+yG_s(f)}{G_n(f)}} dy \tag{I-3}$$

Now, let

$$u = -\left\{ 1 + y \frac{G_s(f)}{G_n(f)} \right\} \frac{G_n(f)}{\sigma^2 G_s(f)}$$

and

$$du = -\frac{dy}{\sigma^2}$$

Then C can be expressed as

$$C = \max_{\{G_s(f) \geq 0\}} -(\ln 2)^{-1} \int_{f_o}^{f_o+W} df \, \exp\left\{ \frac{G_n(f)}{\sigma^2 G_s(f)} \right\} \int_{\infty}^{-\frac{G_n(f)}{\sigma^2 G_s(f)}} \frac{e^u}{u} du \tag{I-4}$$

or

$$C = \max_{\{G_s(f) \geq 0\}} -(\ln 2)^{-1} \int_{f_o}^{f_o+W} \exp\left\{\frac{G_n(f)}{\sigma^2 G_s(f)}\right\} \mathrm{Ei}\left\{\frac{-G_n(f)}{\sigma^2 G_s(f)}\right\} df \qquad \text{(I-5)}$$

which is identical to Eq. (3-7).

APPENDIX II

DERIVATION OF THE OPTIMUM s(t)

To show that the input signal must be from a stationary Gaussian random process for the Rayleigh channel to obtain capacity, consider the following channel.

$$s(t) \longrightarrow \boxed{A} \longrightarrow \bullet \longrightarrow x(t)$$

$G_s(f) = \frac{P}{2W}$

$n(t)$ (into the summing point)
Gaussian
$E[n^2(t)] = N$
$E[n(t)] = 0, \ G_n(f) = \frac{N}{2W} = N_o$

$I[S/X, A = A_1]$ is defined as the average conditional information rate averaged over all $x(t)$, given that $A = A_1$.

To see that the average conditional information rate is a maximum when $s(t)$ is Gaussian, let $A\, s(t = k) \equiv r_k$, be a random variable with power constraint $P' \geqslant A^2\, E[{s_k}^2]$ and $n(t = k) \equiv n_k$, a Gaussian random variable with $E[n_k] = 0$, and $E[{n_k}^2] = N$. Also denote $x(t = k) = x_k$, then

$$E\,({x_k}^2) = E\,({r_k}^2) + E\,({n_k}^2) \leq P' + N \tag{II-1}$$

The average uncertainty of x_k is equal to that of $x(t)$ since $x(t)$ is a stationary process,

$$H(X) = \frac{1}{2}\ \log 2\pi\ \ \epsilon(P' + N) \tag{II-2}$$

This can be shown by considering

$$H_e(X) = -\int_{-\infty}^{\infty} p(x_k)\ \ln\, p(x_k)\ dx_k \tag{II-3}$$

$$H_e(X) - \frac{1}{2} \ln 2\pi e \ [P' + N]$$

$$= - \int_{-\infty}^{\infty} p(x_k) \ \ln p(x_k) + \int_{-\infty}^{\infty} p(x_k) \ln \left\{ \frac{e^{-\frac{x_k^2}{2(P' + N)}}}{[2\pi(P' + N)]^{1/2}} \right\} dx_k \qquad \text{(II-4)}$$

$$= \int_{-\infty}^{\infty} p(x_k) \ln \frac{e^{\frac{-x_k^2}{2(P' + N)}}}{[2\pi(P' + N)]^{1/2} p(x_k)} \quad dx_k$$

Using the fact that $\ln t \leq t-1$, equality if and only if $t = 1$, one obtains

$$H_e(X) - \frac{1}{2} \ \ln 2\pi e(P' + N) \leq \int_{-\infty}^{\infty} dx_k p(x_k) \left[\frac{e^{-x_k^2/2(P' + N)}}{[2\pi(P' + N)]^{1/2} p(x_k)} - 1 \right] \qquad \text{(II-5)}$$

$$= 0$$

equivalently equality if and only if $p(x_k) = \dfrac{e^{-x_k^2/2(P' + N)}}{[2\pi(P' + N)]^{1/2}}$

It has, therefore, been proven that $H(X)$ obtains its maximum when x_k is Gaussian with zero mean and variance $(P' + N)$. If this condition is satisfied

$$H(X) = \frac{1}{2} \ \log 2\pi e(P' + N) \qquad \text{(II-6)}$$

x_k will be Gaussian with zero mean and $E(x_k^2) = P' + N$ if r_k is Gaussian with zero mean and a variance equal to P'. Hence,

$$C = \max_{p(s_k)} I(R/X) = \max_{p(r_k)} \left\{ H(R) - H(R/X) \right\} \tag{II-7}$$

$$= \max_{p(r_k)} \left\{ H(X) - H(X/R) \right\}$$

The maximum $H(X)$ has been obtained above. It must further be shown that for r_k Gaussian distributed, $H(X/R)$ is a minimum.

Solving for $H(X/r=r_k)$

$$H(X/r=r_k) = - \int_{-\infty}^{\infty} dx_k \frac{e^{\frac{-(x_k-r_k)^2}{2N}}}{(2\pi N)^{1/2}} \log \frac{e^{\frac{-(x_k-r_k)^2}{2N}}}{(2\pi N)^{1/2}} \tag{II-8}$$

$$= \frac{1}{2} \log 2\pi e N$$

$$H(X/R) = \int_{-\infty}^{\infty} H(X/r=r_k)\, p(r_k)\, dr_k \tag{II-9}$$

$$= \frac{1}{2} \log 2\pi e N$$

It has, therefore, been shown that r_k and hence s_k must be Gaussian with zero mean and variance $P' = P\sigma^2$ in order for the average conditional rate to be a maximum.

The average information rate is

$$I(S/X) = \int_{-\infty}^{\infty} I(S/X, A = A_1)\, p(A_1)\, dA_1 \tag{II-10}$$

To maximize $I(S/Y)$ the integrand must be maximized for every value of A_1. The integrand is maximized if $I(S/X, A = A_1)$ is maximized for every value of A_1. In order for $I(S/X, A = A_1)$ to be maximized it was proved that $s(t)$ had to be a stationary Gaussian random process with zero mean, and a variance P'.

APPENDIX III

DETERMINING THE LOWER BOUND OF β

Starting with Eq. (3-15') and letting $\alpha = \dfrac{N}{\sigma^2 P_{min}}$

β may be expressed as

$$\beta = -\ln 2 \frac{e^{-\alpha}}{E_1(-\alpha)} \tag{III-1}$$

To find the minimum β calculate

$$\frac{\delta\beta}{\delta\alpha} = \ln 2 \left\{ \frac{e^{-\alpha}}{E_1(-\alpha)} + \frac{e^{-\alpha}}{\alpha^2 E_1(-\alpha)} + \frac{e^{-2\alpha}}{\alpha^2 (E_1(-\alpha))^2} \right\}$$

$$= \ln 2 \frac{e^{-\alpha}[(\alpha+1) E_1(-\alpha) + e^{-\alpha}]}{[\alpha E_1(-\alpha)]^2} \tag{III-2}$$

and note that

$$\frac{\delta\beta}{\delta\alpha} \rightarrow 0 \text{ as } \alpha \longrightarrow \infty$$

Therefore,

$$\beta_{min} = \lim_{\alpha\longrightarrow\infty} -\ln 2 \frac{\dfrac{e^{-\alpha}}{\alpha}}{E_1(-\alpha)} \tag{III-3}$$

Using l'Hopital's rule

$$\beta_{min} = \ln 2 \lim_{\alpha\rightarrow\infty} \frac{\dfrac{e^{-\alpha}}{\alpha} + \dfrac{e^{-\alpha}}{\alpha^2}}{\dfrac{e^{-\alpha}}{\alpha}} = \ln 2 \tag{III-4}$$

This is the result stated in Eq. (3-16).

APPENDIX IV

DERIVATION OF THE HAMMING ERROR RATE EQUATION

1. Glossary of Symbols

This glossary is intended to aid the reader in following the proofs presented by obviating the necessity of searching the Appendix for symbol definition.

Operators, Relationships

$\oplus$: Written $a \oplus b$, where a and b are m binit binary numbers. Treat each of a and b as an m-dimensional vector with elements from the modulo 2 field (Modulo 2 field: contains two elements, 0, 1, with 0+0 = 1+1 = 0, and 0+1, = 1) and add, component by component.

$\sum_{i=1}^{n} \oplus$: Sum, under the conditions of $\oplus$, of n binary numbers.

$\bigcup$ Written $a \bigcup b$, where a and b are sets. $a \bigcup b$ is then the set of all elements belonging to a or b or both.

$\bigcup_{i=1}^{n}$: The set of all elements belonging to one or more of the sets being united.

ϵ: Written $a \epsilon b$, where a is an element of the type found in set b (for example, a itself may be a set, and b a set of sets of the same type as a). Meaning, "a is a member of the set b," or "belongs to."

$\not\epsilon$: See ϵ; " does not belong to."

$P\{a\}$: Probability of a.

$P\{a|b\}$: Probability of a, given b.

<u>Variables</u>

d_j: Binary number, $1 \leq d_j \leq n = 2^m-1$, m binits; member of the set (d_j)

e_j: Binary number, $0 \leq e_j \leq n = 2^m-1$, m binits; the binary representation of the position number (1, ..., n) for a binit in error in a received code word, <u>not</u> including the overall check binit of the DED/SEC case; member of the set (e_j)

e_j': As for e_j, for the code word after the error correction procedure of the decoder has been applied

$l_i(\beta)$: Number of sets (d_j) belonging to $\lambda_i(\beta)$; shown to be constant, $=l_i$, for all β.

l_i: See $l_i(\beta)$.

L_i: Number of sets (d_j) belonging to $\bigcup_{\gamma=1}^{n} \lambda_i(\gamma)$; shown to be equal to nl_i.

m: Length of the binary numbers d_j, e_j, e_j', β and γ.

$m_i(\beta)$: Number of sets (d_j) belonging to $\mu_i(\beta)$; shown to be constant, $=m_i$, for all β.

m_i: See $m_i(\beta)$.

M_i: Number of sets (d_j) belonging to $\bigcup_{\gamma=1}^{n} \mu_i(\gamma)$; shown to be equal to nm_i.

n: Length of a SEC code word, in binits; the number of values that may be assumed by d_j, e_j, e_j', β and γ. (Note that $n = 2^m-1$).

n': Length of a DED/SEC code word; $n = n'-1$ for such codes.

n_e: Number of errors in the first n binits of a received word.

n'_e: Number of errors in the first n binits of a received word after "correction".

n_i: Number of sets (d_j) belonging to ν_i.

n_t: Number of errors in total received word, $= n_e$ for SEC codes, $= n_e$ or n_e-1, SEC/DED codes.

N_i: Same as n_i.

$N_\beta(\lambda_i)$: Number of times β is used as an element in the sets (d_j) belonging to $\bigcup_{\gamma=1}^{n} \lambda_i(\gamma)$; shown to be constant, $N(\mu_i)$, for all β.

$N(\lambda_i)$: See $N_\beta(\lambda_i)$.

$N_\beta(\mu_i)$: Number of times β is used as an element in the sets (d_j) belonging to $\bigcup_{\gamma=1}^{n} \mu_i(\gamma)$; shown to be constant, $N(\mu_i)$, for all β.

$N(\mu_i)$: See $N_\beta(\mu_i)$.

$N_\beta(\nu_i)$: Number of times β is used as an element in the sets (d_j) belonging to ν_i; shown to be constant, $= N(\nu_i)$, for all β.

$N(\nu_i)$: See $N_\beta(\nu_i)$.

P_e: Channel probability of error; for symmetric channels, transitional probability.

y: Omission index. (SEC/DED codes only). $y = 0$ if the received error pattern can be detected but not corrected, and, therefore, the whole code word is discarded; $y = 1$ if the code word is not discarded.

P'_e: Decoder output binit error probability

z: Check index (SEC/DED codes only). $z = 1$ if the overall check binit is received in error, and $z = 0$ if the overall check binit is received correctly (at the decoder input in both cases).

β: Binary number, $1 \; \beta \; n = 2^m-1$, m binits in length.

γ: As for β.

<u>Sets</u>

(d_j): Set of unique binary numbers d_j (see d_j).

(e_j): Set of unique binary numbers e_j, each e_j corresponding to an error in the received code word (see e_j),

(e'_j): Set of unique binary numbers e'_j, each e'_j corresponding to an error in the received, "corrected" code word (see e'_j, e_j).

$\lambda_i(\beta)$: Set of all (d_j), where the number of elements in (d_j) is i satisfying $\sum_{j=1}^{i} \oplus \, d_j = \beta \neq 0$, and $\beta \notin (d_j)$.

$\mu_i(\beta)$: Set of all (d_j), where the number of elements in (d_j) is i, satisfying $\sum_{j=1}^{i} \oplus \, d_j = \beta \epsilon (e_j)$.

ν_i: Set of all (d_j), where the number of elements in (d_j) is i, satisfying $\sum_{j=1}^{i} \oplus \, d_j = 0$.

2. Theorem

For all non-repetitive sets (d_j) of elements d_j, each such element being the binary representation of a number from the set $(1, 2, \ldots, n)$, n = SEC code word length $= 2^m-1$, m an integer, form the sets of sets $\lambda_i(\beta)$, $\mu_i(\beta)$ and ν_i thus:

$\lambda_i(\beta)$ = The set of all sets (d_j) containing i elements and satisfying

$$\sum_{j=1}^{i} {}^{\oplus}\, d_j = \beta \neq 0, \text{ with } \beta\epsilon(d_j);$$

$\mu_i(\beta)$ = The sets (d_j) containing i elements and satisfying

$$\sum_{j=1}^{i} {}^{\oplus}\, d_j = \beta\epsilon(d_j) \qquad \text{(implying } \beta \neq 0 \text{ as well);}$$

ν_i = The set of all sets (d_j) containing i elements and satisfying

$$\sum_{j=1}^{i} {}^{\oplus}\, d_j = 0$$

Define, then, $l_i(\beta)$ = the number of distinct sets $(d_j)\epsilon\ \lambda_i(\beta)$

$m_i(\beta)$ = the number of distinct sets $(d_j)\epsilon\ \mu_i(\beta)$

n_i = the number of distinct sets $(d_j)\epsilon\ \nu_i$

and $N_\beta(\lambda_i)$= the number of times β is used as an element in the distinct sets $(d_j)\epsilon \left[\bigcup_{\gamma=1}^{n} \lambda_i(\gamma)\right]$

$N_\beta(\mu_i)$ = the number of times β is used as an element in the distinct sets $(d_j)\epsilon \left[\bigcup_{\gamma=1}^{n} \mu_i(\gamma)\right]$

$N_\beta(\nu_i)$ = the number of times β is used as an element in the distinct sets $(d_j)\epsilon\nu_i$

Then it is postulated that $l_i(\beta)$, $m_i(\beta)$, and $N_\beta(\lambda_i)$, $N_\beta(\mu_i)$ and $N_\beta(\nu_i)$ are each independent of β, $1 \leq \beta \leq n = 2^m-1$, for a fixed i.

Corollary:

$$m_i = \frac{n-i+1}{n} n_{i-1}$$

$$n_i = \frac{n}{i} l_{i-1}$$

$$l_i = \frac{1}{n} \binom{n}{i} - n_i - m_i$$

$$N(\lambda_i) = i l_i$$

$$N(\mu_i) = i m_i$$

$$N(\nu_i) = \frac{i n_i}{n}$$

where m_i, l_i, $N(\lambda_i)$, $N(\mu_i)$, and $N(\nu_i)$ are the constant values taken on by $m_i(\beta)$, $l_i(\beta)$, $N_\beta(\lambda_i)$, $N_\beta(\mu_i)$ and $N_\beta(\nu_i)$ respectively, for a fixed i and any β, $1 \leq \beta \leq n$.

Proof: The method of proof to be used is mathematical induction. Parts (1) and (5) establish that, if the theorem is true for i - 1, it is then true for i. Part (6) shows the theorem to be true for i = 1, completing the proof.

It is implied throughout the proof that each set (or set of sets) referred to is non-repetitive in its elements (or sets).

Assume, now, that the theorem is true for i-1; the inductive proof follows.

(1) Consider the sets (d_j), $(d_j) \in \nu_{i-1}$; since a given β appears once, at most, in any one (d_j), the number of sets $(d_j) \in \nu_{i-1}$ such that $\beta \notin (d_j)$, is $[n_{i-1}-N_\beta(\nu_{i-1})]$. Adjoining β to each such set results in the formation of $[n_{i-1}-N_\beta(\nu_{i-1})]$ distinct sets (d_j'), each of whose sum $(\oplus)$ is β, and each containing β. Thus, each such $(d_j') \in \mu_i(\beta)$, and

$$m_i(\beta) \geq n_{i-1} - N_\beta(\nu_{i-1})$$

Conversely, for every set $(d_j')\epsilon\ \mu_i(\beta)$, deletion of β from each (d_j') results in a set of distinct sets (d_j), with $(d_j)\epsilon\ \nu_{i-1}$ and $\beta\notin(d_j)$ - i.e.,

$$n_{i-1}-N_\beta(\nu_{i-1})\geq m_i(\beta)$$

Thus, $m_i(\beta) = n_{i-1}-N_\beta(\nu_{i-1})$; since n_{i-1} and $N_\beta(\nu_{i-1})$ are independent of β, $m_i(\beta)$ is similarly independent.

(2) With each $(d_j)\epsilon\ \nu_{i-1}$, containing $(i-1)$ elements, associate the $n-(i-1)$ sets (d_j') formed by adjoining δ_k to (d_j), for each $\delta_k\notin(d_j)$; each (d_j') so formed has the property that $(d_j')\epsilon\ \mu_i(\delta_k)$. Conversely, every $(d_j')\epsilon\ \mu_i(\delta)$ may be associated with exactly one $(d_j)\epsilon\ \nu_{i-1}$ by deletion of δ, and that (d_j) has the property that $\delta\notin(d_j)$.

Thus, associated with each of the $N_\beta(\nu_{i-1})$ sets in ν_{i-1} containing a given β are $(n-i+1)$ unique sets belonging to $\bigcup\limits_{\gamma=1}^{n}\mu_i(\gamma)$ [note that none of these sets can belong to $\mu_i(\beta)$]. Also, with each of the $[n_{i-1}-N_\beta(\nu_{i-1})]$ sets in ν_{i-1} not containing β, there is associated exactly, in one-to-one correspondence, one set in $\bigcup\limits_{\gamma=1}^{n}\mu_i(\gamma)$ [in particular, belong to $\mu_i(\beta)$], such that each such set contains β. Finally, then, the total number of sets in $\bigcup\limits_{\gamma=1}^{n}\mu_i(\gamma)$ containing β is $(n-i+1)N_\beta(\nu_{i-1}) + n_{i-1}-N_\beta(\nu_{i-1})$ -- or,

$$N_\beta(\mu_i) = (n-i)N_\beta(\nu_{i-1}) + n_{i-1}$$

With $N_\beta(\nu_{i-1})$ and n_{i-1} independent of β, $N_\beta(\mu_i)$ is similarly independent.

(3) Consider the sets (d_j') such that $(d_j')\epsilon\ \nu_i$ and $\beta\epsilon(d_j')$ for a given β. By deleting β from each (d_j'), a set of new distinct sets (d_j) is formed, and, for each such (d_j), $\sum\limits_{j=1}^{i-1}{}^{\bullet}\ d_j = \beta$, and $\beta\notin(d_j)$ -- i.e., $(d_j)\epsilon\ \lambda_{i-1}(\beta)$.

Conversely, every $(d_j) \epsilon \lambda_{i-1}(\beta)$ may be associated uniquely with exactly one set $(d_j') = (d_j, \beta)$ in ν_i.

Thus,

$$N_\beta(\nu_i) = l_{i-1}(\beta)$$

Since $l_{i-1}(\beta)$ is independent of β, $N_\beta(\nu_i)$ is similarly independent.

(4) The sets of sets $\lambda_i(\gamma)$, $\mu_i(\gamma)$ and ν_i are disjoint and exhaustive in the set of all sets of i binary numbers chosen out of the binary numbers $(1, 2, \ldots, n = 2^m-1)$ -- that is, any set of i binary numbers (and there are $\binom{n}{i}$ such sets) belongs to exactly one of the $(2n+1)$ sets $\lambda_i(\gamma)$, $1 \leq \gamma \leq n$, $\mu_i(\gamma)$, $1 \leq \gamma \leq n$, and ν_i.

Also, in the set of all i-element sets, each β, $1 \leq \beta \leq n$, is used as an element an equal number of times -- specifically, each β is used $\frac{i}{n}\binom{n}{i}$, or $\binom{n-1}{i-1}$ times. It is established in (2) and (3) that, given a β used $N_\beta(\mu_i)$ times as an element in the sets of $\bigcup_{\gamma=1}^{n} \mu_i(\gamma)$ and $N_\beta(\nu_i)$ times in the sets of ν_i, $N_\beta(\mu_i)$ and $N_\beta(\nu_i)$ are independent of β. It follows that

$$N_\beta(\lambda_i) = \binom{n-1}{i-1} - N_\beta(\mu_i) - N_\beta(\nu_i)$$

with $N_\beta(\ _i)$ independent of β.

(5) Consider a set $(d_j) \epsilon \lambda_i(\beta)$; form a new set $(d_j{}^k)$ by deleting d_k from (d_j), $1 \leq k \leq i$, and adjoining β.

Then, recalling that, in modulo 2 vector arithmetic, $\underline{a} \ominus \underline{b} = \underline{a} \oplus \underline{b}$ ($\ominus$ = vector "subtraction," elements from the modulo 2 field), then

$$\sum_{j=1}^{i}{}^{\oplus} d_j{}^k = \sum_{j=1}^{i}{}^{\oplus} d_j \oplus d_k \oplus \beta = \beta \oplus d_k \oplus \beta = d_k,$$

and $d_k \notin (d_k{}^k)$ - i.e., $(d_j{}^k) \epsilon \lambda_i(d_k)$

Thus, it is demonstrated that, for each $(d_j) \epsilon \lambda_i(\beta)$, there are exactly i associated sets, one each in each of $\lambda_i(d_j)$, 1 j i, containing β; it is obvious that no two different $(d_j) \epsilon \lambda_i(\beta)$ can be so associated with the same $(d_j^{\,k})$ in some $\lambda_i(d_k)$, and that every $(d_j') \epsilon \lambda_i(\delta)$ with $\beta \epsilon (d_j')$ must be associated with exactly one $(d_j) \epsilon \lambda_i(\beta)$. Since the total number of sets in $\bigcup_{\gamma=1}^{n} \lambda_i(\gamma)$ containing β is $N_\beta(\lambda_i)$, the number of times β is used as an element of these sets, it follows that the number of sets in $\lambda_i(\beta)$ is given by

$$l_i(\beta) = \frac{N_\beta(\lambda_i)}{i} \quad , \; i \neq 0$$

Since $N_\beta(\lambda_i)$ is independent of β, similarly $l_i(\beta)$ is independent of β.

(6) Finally, consider the case for i = 1. Then every set (d_j) consists of a single element d_1. For every such set, $\sum_{j=1}^{i}{}^{\oplus} d_j = d_1 \, \epsilon (d_j)$; thus every set $(d_j) \in \mu_1(d_j)$, and $m_1(\beta) = 1$, for all β, $1 \leq \beta \leq n$. Also, $l_i(\beta) = 0$ and $N_\beta(\lambda_i) = N_\beta(\nu_i) = 0$, $N_\beta(\mu_i) = 1$, independent of β.

(7) Since $l_i(\beta)$, $m_i(\beta)$, $N_\beta(\lambda_i)$, $N_\beta(\mu_i)$ and $N_\beta(\nu_i)$ are all independent of the β chosen, redefine

$$l_i = l_i(\beta); \; m_i = m_i(\beta); \text{ and}$$

$$N_\beta(\lambda_i) = N(\lambda_i); \; N_\beta(\mu_i) = N(\mu_i); \; N_\beta(\nu_i) = N(\nu_i).$$

Summarizing then, it has been shown that

$$\begin{aligned}
m_i &= n_{i-1} - N(\nu_{i-1}) \\
N(\mu_i) &= (n-i)N(\nu_{i-1}) + n_{i-1} \\
N(\nu_i) &= l_{i-1} \\
N(\lambda_i) &= \binom{n-1}{i-1} - N(\mu_i) - N(\nu_i) \\
l_i &= N(\lambda_i)/i
\end{aligned} \qquad \text{(IV-1)}$$

Also, in the set ν_i of all i-length sets whose sum is zero, the total number of elements is in_i; but each of the n elements (1, ..., n) is used an equal number of times, so that the number of times any one element appears in these sets is given by

$$N(\nu_i) = \frac{in_i}{n} \tag{IV-2}$$

Using (IV-1) with (IV-2), it follows that

$$m_i = \frac{(n-i+1)n_{i-1}}{n} \tag{IV-3}$$

$$n_i = \frac{n\ l_{i-1}}{i} \tag{IV-4}$$

$$l_i = \frac{1}{n}\left[\binom{n}{i} - n_i\right] - m_i \tag{IV-5}$$

$$N(\lambda_i) = i\ l_i \tag{IV-6}$$

$$N(\mu_i) = i\ m_i \tag{IV-7}$$

$$N(\nu_i) = \frac{i\ n_i}{n} \tag{IV-8}$$

(9) The following lemma results immediately:

<u>Lemma</u>

The probability that an arbitrary non-repetitive set (d_j) of i binary numbers chosen from the set (1, 2, ..., $n=2^m-1$) is such that

(a) $(d_j)\epsilon\lambda_i(\beta)$, is given by $l_i/\binom{n}{i}$;

(b) $(d_j)\epsilon\mu_i(\beta)$, is given by $m_i/\binom{n}{i}$;

(c) $(d_j)\epsilon\nu_i$, is given by $n_i/\binom{n}{i}$.

3. <u>Error Rate Equation, SEC Hamming Codes</u>

For these codes, the length $n = 2^m-1$. Then,

Probability that the β^{th} binit is in error after correction =

$$P\{\beta\epsilon(e_j')\} = \sum P(\beta\epsilon(e_j')\,|n_e=i)P\{n_e=i\} \qquad \text{(IV-9)}$$

(1)

$$P\{\beta\epsilon(e_j')\,|n_e=1\} = P\left\{\beta\epsilon(e_j), \sum\oplus\; e_j \neq \beta|n_e=1\right\}$$

$$+ P\left\{\beta\not\epsilon(e_j), \sum\oplus\; e_j = \beta\;|n_e=1\right\} \qquad \text{(IV-10)}$$

$$= P\left\{\beta\epsilon(e_j), (e_j)\epsilon\left\{\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_1(\gamma)\right]\cup\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \lambda_1(\gamma)\right]\cup\nu_1\right\}|n_e=1\right\}$$

$$+ P\{(e_j)\epsilon\;\lambda_1(\beta)|n_e=1\}$$

-- with the $\lambda_1(\gamma)$, $\mu_1(\gamma)$ and ν_1 all disjoint, this becomes

$$P\{\beta\epsilon(e_j')|n_e=1\} = P\{\beta\epsilon(e_j),\;(e_j)\epsilon\;\nu_1\;|n_e=1\}$$

$$+P\left\{\beta\epsilon(e_j),\;(e_j)\epsilon\;\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_1(\gamma)\right]|n_e=1\right\}$$

$$+P\left\{\beta\epsilon(e_j),\;(e_j)\epsilon\;\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \lambda_1(\gamma)\right]\;|n_e=1\right\} + P\{(e_j)\epsilon\;\lambda_1(\beta)\;|n_e=1\} \qquad \text{(IV-11)}$$

or

$$P\{\beta\epsilon(e_j')\,|n_e=1\} = P\{\beta\epsilon(e_j)|\;(e_j)\epsilon\;\nu_1,\;n_e=1\}\;P\{\;(e_j)\epsilon\;\nu_1\;|n_e=1\}$$

$$+P\left\{\beta\epsilon(e_j)|\;(e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_1(\gamma)\right],\;n_e=1\right\}P\left\{(e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_1(\gamma)\right]|n_e=1\right\}$$

$$+P\left\{\beta\epsilon(e_j) \mid (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \lambda_i(\gamma)\right], n_e=i\right\} P\left\{(e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \lambda_i(\gamma)\right] \mid n_e=i\right\}$$

$$+ P\{(e_j)\epsilon\ \lambda_i(\beta)|n_e=i\} \qquad \text{(IV-12)}$$

(2)

$$P\left\{(e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_i(\gamma)\right]|n_e=i\right\} = \frac{\text{Number of } (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_i(\gamma)\right]}{\text{Total number of } (e_j)} \qquad \text{(IV-13)}$$

with the $\mu_i(\gamma)$ disjoint,

$$\text{Number of } (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_i(\gamma)\right] = \sum_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \text{Number of } (e_j)\epsilon\ \mu_i(\gamma)$$

$$= \sum_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} m_i(\gamma)$$

-- but $m_i(\gamma) = m_i$, independent of γ . The numerator then becomes $(n-1)m_i$
The denominator of (IV-13) is merely $\binom{n}{i}$; thus,

$$P\left\{(e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_i(\gamma)\right]|n_e=i\right\} = \frac{(n-1)m_i}{\binom{n}{i}} \qquad \text{(IV-14)}$$

Similarly,

$$P\left\{(e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} {}_i(\gamma)\right] |n_e= i\right\} = \frac{(n-1)l_i}{\binom{n}{i}} \qquad \text{(IV-15)}$$

$$P\ \{\ (e_j)\epsilon\ \nu_i(\beta)|n_e= i\} = \frac{n_i}{\binom{n}{i}} \qquad \text{(IV-16)}$$

and

$$P\{\ (e_j)\epsilon\ \lambda_i(\beta)|n_e= i\} = \frac{l_i}{\binom{n}{i}} \qquad \text{(IV-17)}$$

(3)

$$P\left\{\beta\epsilon\ (e_j) \mid (e_j)\ \epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_i(\gamma)\right],\ n_e = i\right\}$$

$$= \frac{\text{Number of } (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_i(\gamma)\right] \text{ such that } \beta\epsilon(e_j)}{\text{Total number of } (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_i(\gamma)\right]} \tag{IV-18}$$

Now,

$$\text{Number of } (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_i(\gamma)\right] \text{ such that } \beta\epsilon(e_j)$$

$$= \text{Number of } (e_j)\epsilon\left[\bigcup_{\gamma=1}^{n} \mu_i(\gamma)\right], \text{ with } \beta\epsilon(e_j)$$

$$- \text{ Number of } (e_j)\epsilon\mu_i(\beta) \text{ with } \beta\epsilon(e_j). \tag{IV-19}$$

Number of $(e_j)\epsilon\left[\bigcup_{\gamma=1}^{n} \mu_i(\gamma)\right]$ with $\beta\epsilon(e_j) = N(\mu_i)$, while all $(e_j)\epsilon\mu_i(\beta)$, $(=m_i)$ satisfy $\beta\epsilon(e_j)$. Thus, the numerator of (IV-18) becomes

$$N(\mu_i) - m_i = (i-1)m_i$$

Again, since the $\mu_i(\gamma)$ are disjoint,

$$\text{Total number of } (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \mu_i(\gamma)\right] = \sum_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} \text{Number of } (e_j)\epsilon\mu_i(\gamma)$$

$$= \sum_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n} m_i(\gamma) = (n-1)m_i \tag{IV-20}$$

So that

$$P\left\{ \beta\epsilon(e_j) \middle| (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1 \\ (\gamma\neq\beta)}}^{n} \mu_i(\gamma) \right] , n_e = i \right\} = \frac{i-1}{n-1} \tag{IV-21}$$

Similarly,

$$P\left\{ \beta\epsilon(e_j) \middle| (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1 \\ (\gamma\neq\beta)}}^{n} \lambda_i(\gamma) \right] , n_e = i \right\}$$

$$= \frac{\text{Number of } (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1 \\ (\gamma\neq\beta)}}^{n} \lambda_i(\gamma) \right] \text{ such that } \beta\epsilon(e_j)}{\text{Total number of } (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1 \\ (\gamma\neq\beta)}}^{n} \lambda_i(\gamma) \right]} \tag{IV-22}$$

or

$$P\left\{ \beta\epsilon(e_j) \middle| (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1 \\ (\gamma\neq\beta)}}^{n} \lambda_i(\gamma) \right] , n_e = i \right\} = \frac{N(\lambda_i)}{(n-1)l_i} = \frac{i}{n-1} \tag{IV-23}$$

since for every $(e_j)\epsilon\lambda_i(\beta)$, $\beta\notin(e_j)$;

also,

$$P\{\beta\epsilon(e_j)|(e_j)\epsilon\nu_i,\ n_e = i\} = \frac{N(\nu_i)}{n_i} = \frac{i}{n} \tag{IV-24}$$

Substituting, (IV-12) becomes

(4)

$$P\{\beta\epsilon(e_j')|n_e = i\} = \frac{i}{n} \cdot \frac{n_i}{\binom{n}{i}} + \frac{i-1}{n-1} \cdot \frac{(n-1)m_i}{\binom{n}{i}} + \frac{i}{n-1} \cdot \frac{(n-1)l_i}{\binom{n}{i}} + \frac{l_i}{\binom{n}{i}}$$

$$= \frac{1}{\binom{n}{i}} \left\{ \frac{in_i}{n} + (i-1)m_i + (i+1)\, l_i \right\} . \tag{IV-25}$$

Define: $L_i = nl_i$

$$M_i = nm_i \tag{IV-26}$$

$N_i = n_i$

Then

$$P\{\beta\epsilon(e_j')|n_e = i\} = \frac{1}{n\binom{n}{i}}\left[\,(i+1)\,L_i + iN_i + (i-1)\,M_i\right] \tag{IV-27}$$

and the parameters are given by

$M_i = (n-i+1)\,N_{i-1}$

$$N_i = \frac{1}{i}\,L_{i-1} \tag{IV-28}$$

$L_i = \binom{n}{i} - N_i - M_i$

with initial values $M_1 = n$, $L_1 = N_1 = 0$

(or, alternatively, $M_0 = L_0 = 0$, $N_o = 1$).

(5)

$P\{n_e = i\} = \binom{n}{i} P_e^{\,i}(1-P_e)^{n-i}$, where P_e = channel probability of error.

Equation (IV-9) becomes

$$P\{\beta\epsilon(e_j')\} = \sum_{i=0}^{n} \frac{1}{n}\left[(i+1)L_i + iN_i + (i-1)M_i\right] P_e^{\,i}(1-P_e)^{n-i} \tag{IV-29}$$

(6)

P_e' = probability that an arbitrary information binit is in error at the decoder output = $\sum\limits_{\substack{\text{info bits}\\\text{in the code}\\\text{word}}}$ (Prob. that the arbitrary binit = a specific info. binit in the code word) (Prob. that the specific info. binit is in error).

Since the probability that any specific binit β in the code word is in error after correction, $P\{\beta\epsilon(e_j')\}$, is independent of β, this is also the probability that any info. bit is in error. Hence,

$P_e' = P\{\beta\epsilon(e_j')\}$

or,

$$P_e' = \frac{1}{n} \sum_{i=0}^{n} \Big[(i-1)\, M_i + iN_i + (i+1)L_i\Big] P_e^{\,i}(1-P_e)^{n-i} \qquad \text{(IV-30)}$$

In actual computation, the coefficients of the terms for $i = 0$, $i = 1$ are both zero; the summation may start at $i = 2$.

4. Error Rate Equation; SEC/DED Hamming Codes

For these codes, P_e' is defined as the probability that an arbitrary information binit is in error after decoding given that the code word was not discarded by the decoder. Then, the desired probability, analagous to 3., is

Probability that the β^{th} binit is in error after correction, given that the code word is not discarded =

$$P\{\beta\ (e_j')|y = 1\} = \frac{1}{P\{y=1\}} P\{\beta\ (e_j'),\ y = 1\} \qquad \text{(IV-31)}$$

$$= \frac{1}{P\{y=1\}} \left\{ \sum_{i=0}^{n} P\{\beta\beta\ (e_j'),\ y = 1,\ z = 0,\ n_t = i\} + \sum_{i=1}^{n+1} P\{\beta\ (e_j'),\ y = 1,\ z = 1,\ n_t = i\} \right\}$$

where the length of the code word is $n = n+1 = 2^m$; $z=0/z=1$ indicate that the overall check binit is not/is in error, respectively, and $y=0/y=1$ indicate that the code word is/is not discarded.

Again,

$$P\{\beta\ (e_j')\ y=1\} = \frac{1}{P\{y=1\}} \left\{ \sum_{i=0}^{n} P\{\beta\ (e_j'),\ y=1|n_t=i,\ z=0\}\ P\{z=0|n_t=i\}\ P\{n_t=i\} + \sum_{i=1}^{n+1} P\{\beta\ (e_j'),\ y=1|n_t=i,\ z=1\}\ P\{z=1|n_t=i\}\ P\{n_t=i\} \right\} \qquad \text{(IV-32)}$$

(1)

For a code word to be discarded, two sets of conditions may be imposed.

(a) The overall parity check is zero, while the check word is not. This detects and discards all double errors, as well as the majority of patterns of all other even numbers of errors.

(b) The conditions of (a) are satisfied, <u>or</u> the check word is zero while the overall check is not. This detects and discards a large number of error patterns with odd, >1, errors; however, it will also discard the single one-error pattern in which the error occurs in the overall check binit.

The error rate equation under conditions (a) is developed in detail, while that for (b) is stated without proof.

(2)

$$P\{\beta\epsilon(e_j'),\ y = 1|n_t = i,\ z = 0\}$$

$$= P\left\{\beta\epsilon(e_j),\ \sum\nolimits^{\oplus} e_j \neq \beta,\ y = 1|n_t = i,\ z = 0\right\}$$

$$+ P\left\{\beta\not\epsilon(e_j),\ \sum\nolimits^{\oplus} e_j = \beta,\ y = 1|n_t = i,\ z = 0\right\}$$

-- with $n_t = i$ and $z = 0$, then $n_e = i$. (IV-33)

Following the method of 3.(1), this becomes

$$P\{\beta\epsilon(e_j'),\ y = 1|n_t = i,\ z = o\}$$

$$= P\{\beta\epsilon(e_j),\ (e_j)\epsilon\nu_i,\ y = 1|n_e = i\}$$

$$+ P\left\{\beta\epsilon(e_j),\ (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1 \\ (\gamma\neq\beta)}}^{n}\mu_i(\gamma)\right],\ y = 1|n_e = i\right\}$$

$$+ P\left\{\beta\epsilon(e_j),\ (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1 \\ (\gamma\neq\beta)}}^{n}\lambda_i(\gamma)\right],\ y = 1|n_e = i\right\}$$

$$+ P\{(e_j)\epsilon\lambda_i(\beta),\ y = 1|n_e = i\} \qquad \text{(IV-34)}$$

For odd n_e, $y = 1$; for even n_e, however, $y = 1$ and $(e_j)\epsilon\mu_{n_e}(\gamma)$ or $\lambda_{n_e}(\gamma)$ for some γ cannot both occur simultaneously. For $(e_j)\epsilon\nu_{n_e}$, $y = 1$ for all n_e, even and odd.

Thus,

$$P\{\beta\epsilon(e_j'),\ y = 1|n_t = i,\ z = 0\} = P\{\beta\epsilon(e_j),\ (e_j)\epsilon\nu_i|n_e = i\},\ i \text{ even} \qquad \text{(IV-35A)}$$

$$= P\{\beta\epsilon(e_j),\ (e_j)\epsilon\nu_i|n_e = i\}$$

$$+ P\left\{\beta\epsilon(e_j),\ (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1 \\ (\gamma\neq\beta)}}^{n}\mu_i(\gamma)\right]|n_e = i\right\}$$

$$+ P\left\{\beta\epsilon(e_j),\ (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1 \\ (\gamma\neq\beta)}}^{n}\lambda_i(\gamma)\right]|n_e = i\right\}$$

$$+ P\{(e_j)\epsilon\lambda_i(\beta)|n_e = i\},\ i \text{ odd} \qquad \text{(IV-35B)}$$

(3) Now,

$$P\{\beta\epsilon(e_j),\ (e_j)\epsilon\nu_i\ n_e = i\} = P\{\beta\epsilon(e_j)|(e_j)\epsilon\nu_i,\ n_e = i\}\ P\{(e_j)\epsilon\nu_i|n_e = i\} \tag{IV-36}$$

$$P\left\{\beta\epsilon(e_j),\ (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n}\mu_i(\gamma)\right]|n_e = i\right\}$$

$$= P\left\{\beta\epsilon(e_j)|(e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n}\mu_i(\gamma)\right],\ n_e = i\right\} P\left\{(e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n}\mu_i(\gamma)\right]|n_e = i\right\}; \tag{IV-37}$$

and

$$P\left\{\beta\epsilon(e_j),\ (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n}\lambda_i(\gamma)\right]|n_e = i\right\}$$

$$= P\left\{\beta\epsilon(e_j)|(e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n}\lambda_i(\gamma)\right],\ n_e = i\right\} P\left\{(e_j)\epsilon\left[\bigcup_{\substack{\gamma=1\\(\gamma\neq\beta)}}^{n}\lambda_i(\gamma)\right]|n_e = i\right\}. \tag{IV-38}$$

Using these, with (IV-14) through (IV-23), (IV-35A, B) become

$$P\{\beta\epsilon(e_j'),\ y = 1|n_t = i,\ z = 0\} = \frac{in_i}{n\binom{n}{i}},\ i \text{ even} \tag{IV-39A}$$

$$P\{\beta\epsilon(e_j'),\ y = 1|n_t = i,\ z = 0\} = \frac{1}{\binom{n}{i}}\left\{\frac{in_i}{n} + (i-1)m_i + (i+1)l_i\right\},\ i \text{ odd} \tag{IV-39B}$$

(4)

In a similar manner, with $n_t = i$, $z = 1$ (then $n_e = i-1$),

$$P\{\beta\epsilon(e_j'),\ y = 1|n_t = i,\ z = 1\}$$

$$= P\{\beta\epsilon(e_j),\ (e_j)\epsilon\nu_{i-1},\ y = 1|n_e = i-1\}$$

$$+ P\left\{\beta\epsilon(e_j),\ (e_j)\epsilon\left[\bigcup_{\substack{\gamma=1 \\ (\gamma\neq\beta)}}^{n} \mu_{i-1}(\gamma)\right],\ y = 1 \middle| n_e = i-1\right\}$$

$$+ P\{(e_j)\epsilon\lambda_{i-1}(\beta),\ y = 1|n_e = i-1\} \qquad \text{(IV-40)}$$

obtaining

$$P\{\beta\epsilon(e_j'),\ y = 1|n_t = i,\ z = 1\}$$

$$= \frac{(i-1)n_{i-1}}{n\binom{n}{i-1}},\ i \text{ even} \qquad \text{(IV-41A)}$$

$$= \frac{1}{\binom{n}{i-1}}\left\{\frac{(i-1)n_{i-1}}{n} + (i-2)m_{i-1} + il_{i-1}\right\},\ i \text{ odd} \qquad \text{(IV-41B)}$$

(5)

For an arbitrary received error pattern containing i errors,

$$P\{z = 0|n_t = i\} = \frac{n'-i}{n'} = \frac{n-i+1}{n+1} \qquad \text{(IV-42A)}$$

and

$$P\{z = 1|n_t = i\} = \frac{i}{n+1} \qquad \text{(IV-42B)}$$

Also,

$$P\{n_t = i\} = \binom{n+1}{i} P_e^{\,i} (1-P_e)^{n+1-i} \qquad \text{(IV-43)}$$

(6)

Finally, substituting (IV-39A, B) through (IV-43) into (IV-32),

$$P\{\beta\epsilon(e_j')|y = 1\} = \frac{1}{P\{y=1\}}\left\{\sum_{\substack{i=0 \\ (i \text{ even})}}^{n-1} \left[\frac{in_i}{n\binom{n}{i}}\right]\left[\frac{n-i+1}{n+1}\right]\binom{n+1}{i} P_e^{\,i}(1-P_e)^{n+1-i}\right.$$

$$+ \sum_{\substack{i=1 \\ (i\ \text{odd})}}^{n} \frac{1}{\binom{n}{i}} \left[\frac{in_i}{n} + (i-1)m_i + (i+1)l_i \right] \left[\frac{n-i+1}{n+1} \right] \binom{n+1}{i} P_e^{\,i}(1-P_e)^{n+1-i}$$

$$+ \sum_{\substack{i=1 \\ (i\ \text{odd})}}^{n} \frac{1}{\binom{n}{i-1}} \left[\frac{(i-1)n_{i-1}}{n} + (i-2)m_{i-1} + (i)_{i-1} \right] \left[\frac{i}{n+1} \right] \binom{n+1}{i} P_e^{\,i}(1-P_e)^{n+1-i}$$

$$+ \sum_{\substack{i=2 \\ (i\ \text{even})}}^{n+1} \left[\frac{(i-1)n_{i-1}}{n\binom{n}{i-1}} \right] \left[\frac{i}{n+1} \right] \binom{n+1}{i} P_e^{\,i}(1-P_e)^{n+1-i} \Bigg\} \qquad \text{(IV-44)}$$

Now,

$$\binom{n+1}{i} = \frac{(n+1)!}{i!(n+1-i)!} = \left[\frac{n+1}{n+1-i} \right] \left[\frac{n!}{i!(n-i)!} \right] = \left[\frac{n+1}{n+1-i} \right] \binom{n}{i}$$

$$= \left[\frac{n+1}{i} \right] \frac{n!}{(i-1)![n-(i-1)]!!} = \left[\frac{n+1}{i} \right] \binom{n}{i-1} \qquad \text{(IV-45)}$$

so that

$$P\{\beta\epsilon(e_j')|y=1\} = \frac{1}{P\{y=1\}} \left\{ \frac{1}{n} \sum_{\substack{i=0 \\ (i\ \text{even})}}^{n-1} in_i\, P_e^{\,i}(1-P_e)^{n+1-i} \right.$$

$$+ \frac{1}{n} \sum_{\substack{i=1 \\ (i\ \text{odd})}}^{n} [in_i + (i-1)nm_i + (i+1)nl_i]\, P_e^{\,i}(1-P_e)^{n+1-i}$$

$$+ \frac{1}{n} \sum_{\substack{i=1 \\ (i\ \text{odd})}}^{n} [(i-1)n_{i-1} + (i-2)nm_{i-1} + inl_{i-1}]P_e^{\,i}(1-P_e)^{n+1-i}$$

$$\left. + \frac{1}{n} \sum_{\substack{i=2 \\ (i\ \text{even})}}^{n+1} (i-1)n_{i-1}\, P_e^{\,i}(1-P_e)^{n+1-i} \right\} \qquad \text{(IV-46)}$$

Using (IV-37),

$$P\{\beta\epsilon(e_j') \; y = 1\} = \frac{1}{P\{y=1\}n} \left\{ \sum_{\substack{i=0 \\ (i \text{ even})}}^{n-1} iN_i \, P_e^{\,i}(1-P_e)^{n+1-i} \right.$$

$$+ \sum_{\substack{i=1 \\ (i \text{ odd})}}^{n} [iN_i + (i-1)M_i + (i+1)L_i] \;\; P_e^{\,i}(1-P_e)^{n+1-i}$$

$$+ \sum_{\substack{i=1 \\ (i \text{ odd})}}^{n} [(i-1)N_{i-1} + (i-2)M_{i-1} + iL_{i-1}] \;\; P_e^{\,i}(1-P_e)^{n+1-i}$$

$$\left. + \sum_{\substack{i=2 \\ (i \text{ even})}}^{n+1} (i-1)N_{i-1} \, P_e^{\,i}(1-P_e)^{n+1-i} \right\} \qquad \text{(IV-47)}$$

(7)

Now,

$$\sum_{\substack{i=0 \\ (i \text{ even})}}^{n-1} iN_i \, P_e^{\,i}(1-P_e)^{n+1-i} = \sum_{\substack{i=2 \\ (i \text{ even})}}^{n+1} iN_i \, P_e^{\,i}(1-P_e)^{n-i} \qquad \text{(IV-48)}$$

-- since the first term in the left summation is zero. For N_{n+1}, note that, although it is implied that the L_i, M_i, and N_i are defined only for $0 \leq i \leq n$, setting $\binom{n}{i} = 0$ for $i < 0$, $i > n$, allows the iterative equations (IV-28) for L_i, M_i, and N_i to be extended to values of $i > n$; a similar situation exists for the rewritten forms

$$N_{i-1} = \frac{M_i}{n-i+1} \quad ;$$

$$L_{i-1} = iN_i \quad ;$$

$$M_{i-1} = \binom{n}{i} - L_{i-1} - N_{i-1} \quad ;$$

for $i < 0$; a brief examination reveals that the values obtained for L_i, M_i, and N_i are identically zero for $i < 0$ and $i > n$. In particular, $N_{n+1} = 0$, allowing the extension of summation of (IV-48).

Then, (IV-47) becomes

$$P\{\beta\epsilon(e_j')|y = 1\} = \frac{1}{nP\{y=1\}}\left\{\sum_{\substack{i=2 \\ (i\ \text{even})}}^{n+1} [iN_i + (i-1)N_{i-1}]\ P_e^{\,i}(1-P_e)^{n+1-i}\right.$$

$$\left.+\sum_{\substack{i=1 \\ (i\ \text{odd})}}^{n} [(i-2)M_{i-1} + (i-1)M_i + (i-1)N_{i-1} + iN_i + iL_{i-1} + (i+1)L_i\ ]P_e^i(1-P_e)^{n+1-i}\right\} \quad \text{(IV-50)}$$

(8)

An argument identical to that of 3.(6) results in the conclusion that

$$P_e' = P\ \{\text{arbitrary info. bit in error after decoding}\}$$

$$= P\{\beta\epsilon(e_j')|y = 1\} \quad \text{(IV-51)}$$

or

$$P_e' = \frac{1}{nP\{y=1\}}\left\{\sum_{\substack{i=2 \\ (i\ \text{even})}}^{n+1} [iN_i + (i-1)N_{i-1}]\ P_e^{\,i}(1-P_e)^{n+1-i}\right.$$

$$\left.+\sum_{\substack{i=1 \\ (i\ \text{odd})}}^{n} [(i-2)M_{i-1}+(i-1)M_i+(i-1)N_{i-1}+iN_i+iL_{i-1}+(i+1)L_i]\ P_e^i(1-P_e)^{n+1-i}\right\} \quad \text{(IV-52)}$$

This may be reduced slightly by using Eq. (IV-28) if desired; however, the present symmetrical form is illustrative of the principles involved, and is convenient for computer programming.

(9)

$$P\{y = 1\} = \sum_{i=0}^{n+1} P\{y = 1|n_t = i\}\ P\{n_t = i\} \quad \text{(IV-53)}$$

Now, $P\{y = 1|n_t = i\} = 1$, i odd ;

but $P\{\ y = 1|n_t = i\} = 0$, i even and $\sum\oplus\ e_j \neq 0$; (IV-54)

Thus, <u>for i even,</u>

$$P\{y = 1|n_t = i\} = P\{y = 1,\ z = 0|n_t = i\} + P\{y = 1,\ z = 1|n_t = i\}$$
$$= P\{y = 1|n_t = i,\ z = 0\}\ P\{z = 0|n_t = i\}$$
$$+ P\{y = 1|n_t = i,\ z = 1\}\ P\{z = 1|n_t = i\}$$
$$= P\{(e_j)\epsilon\nu_i\ n_e = i\}\ P\{z = 0|n_t = i\}$$
$$+ P\{(e_j)\epsilon\nu_{i-1}|n_e = i-1\}\ P\{z = 1|n_t = i\} \qquad \text{(IV-55)}$$

Substituting from (IV-16) and (IV-42A, B), this becomes

$$P\{y = 1|n_t = i\} = \left[\frac{n_i}{\binom{n}{i}}\right]\left[\frac{n-i+1}{n+1}\right] + \left[\frac{n_{i-1}}{\binom{n}{i-1}}\right]\left[\frac{i}{n+1}\right]$$

$$= \frac{n_i}{\binom{n}{i}\frac{n+1}{n+1-i}} + \frac{n_{i-1}}{\binom{n}{i-1}\frac{n+1}{i}}\ ,\ i \text{ even}$$

Using (IV-45)

$$P\{y = 1|n_t = i\} = \frac{1}{\binom{n+1}{i}}\ [n_i + n_{i-1}]\ ,\ i \text{ even} \qquad \text{(IV-56)}$$

(10)

Using (IV-55), (IV-56), and (IV-43), (IV-53) becomes

$$P\{y = 1\} = \sum_{\substack{i=1\\(i\ \text{odd})}}^{n} \binom{n+1}{i} P_e^i(1-P_e)^{n-i+1} + \sum_{\substack{i=0\\(i\ \text{even})}}^{n+1} [N_i + N_{i-1}]\ P_e^i(1-P_e)^{n-i+1} \qquad \text{(IV-57)}$$

where, as before, $N_j = 0,\ j<0,\ j>n$.

(11)

A similar development for conditions (b) results in the following expressions:

$$P'_e = \frac{1}{nP\{y=1\}} \left\{ \sum_{\substack{i=2 \\ (i \text{ even})}}^{n+1} [iN_i + (i-1)N_{i-1}] \, P_e^{\,i} (1-P_e)^{n+1-i} \right.$$

$$\left. + \sum_{\substack{i=1 \\ (i \text{ odd})}}^{n} [(i-2)M_{i-1} + (i-1)M_i + iL_{i-1} + (i+1)L_i] \, P_e^{\,i} (1-P_e)^{n+1-i} \right\} \qquad \text{(IV-58)}$$

and

$$P\{y = 1\} = \left\{ \sum_{\substack{i=1 \\ (i \text{ odd})}}^{n} [L_i + L_{i-1} + M_i + M_{i-1}] P_e^{\,i} (1-P_e)^{n+1-i} \right.$$

$$\left. + \sum_{\substack{i=0 \\ (i \text{ even})}}^{n+1} [N_i + N_{i-1}] \, P_e^{\,i} (1-P_e)^{n+1-i} \right\} \qquad \text{(IV-59)}$$

APPENDIX V

RESULTS OF COMPUTER SIMULATION -- BOSE-CHANDHURI (15,7) AND (15,5) CODES

The tables included in this appendix show the numerical results of computer simulation of the Bose-Chandhuri (15,7) 2-error correcting code and the (15,5) 3-error correcting code. The tables are arranged so that the entry in the i^{th} column and j^{th} row is the number of i-weight error patterns resulting in j errors in the decoded information binits. The coefficient of the i^{th} term in the corresponding error rate equation is determined by

$$\frac{1}{k} \sum_{j=0}^{k} j \, B_{ij}$$

, where B_{ij} is the i column, j row entry, and k is the total number of information binits in the code word ($k = 7$ for the (15,7) code; $k = 5$ for the (15, 5) code).

j \ i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	15	105	135	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	65	273	364	471	362	218	41	0	0	0	0	0	0
2	0	0	0	105	387	891	1176	1211	966	474	135	29	0	0	0	0
3	0	0	0	105	452	1083	1635	1969	1709	1208	530	224	45	0	0	0
4	0	0	0	45	224	530	1208	1709	1969	1635	1083	452	105	0	0	0
5	0	0	0	0	29	135	474	966	1211	1176	891	387	105	0	0	0
6	0	0	0	0	0	0	41	218	362	471	364	273	65	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	135	105	15	1

Table V-1. Computer Simulation, B-C (15,7) Code

j \ i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	15	105	455	420	28	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	161	616	984	1083	1062	721	339	154	0	0	0	0
2	0	0	0	0	322	1147	1773	2166	2124	1527	873	308	0	0	0	0
3	0	0	0	0	308	873	1527	2124	2166	1773	1147	322	0	0	0	0
4	0	0	0	0	154	339	721	1062	1083	984	616	161	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	28	420	455	105	15	1

Table V-2. Computer Simulation, B-C (15,5) Code

APPENDIX VI

HAMMING CODE ERROR RATE EQUATION COEFFICIENTS

This appendix contains tables of the error rate equation coefficients for the Hamming SEC codes, and the SEC/DED codes operating under the two previously postulated sets of word rejection conditions as calculated by the IBM 7090 Digital Computer.

Although each coefficient is an integer, the size of most of the coefficients is beyond the integer storage capabilities of most computers. For this reason, the coefficients are presented in the form

X.XXXXXXXEXXX

where the symbolic "EXXX" is to be read as "x 10^{+XXX}". Again, as a result of the characteristics of the computer, numbers such as "7.0000000×10^{0}" often appear as "6.9999999E 0".

The coefficients presented are "A(I)", to be read as "a_i", where

$$P_e' = \sum_{i=0}^{n} a_i \, P_e^{\,i}(1-P_e)^{n-i}, \text{ for SEC codes;}$$

and $P_e' = \dfrac{P\{\text{arbitrary information binit in error and word accepted}\}}{P\{\text{word accepted}\}}$

with

$$P\{\text{arbitrary information binit in error } \underline{\text{and}} \text{ word accepted}\} = \sum_{i=0}^{n'} a_i \, P_e^{\,i}(1-P_e)^{n'-i}, \text{ for SEC/DED codes;}$$

and "B(I)", to be read as "b_i", where

$$P\{\text{word accepted}\} = 1 - P\{\text{word rejected}\} = 1 - \left[\sum_{i=0}^{n'} b_i \, P_e^{\,i}(1-P_e)^{n'-i}\right], \text{ for}$$

SEC/DED codes.

ERROR RATE EQUATION COEFFICIENTS

HAMMING SINGLE ERROR CORRECTING CODES

I	A(I)	I	A(I)	I	A(I)
N = 7					
0	0. E 0	3	1.8999999E 1	6	6.9999999E 0
1	0. E 0	4	1.5999999E 1	7	1.0000000E 0
2	8.9999999E 0	5	1.2000000E 1		
N = 15					
0	0. E 0	6	2.0929997E 3	11	9.7299995E 2
1	0. E 0	7	3.0669997E 3	12	3.3599998E 2
2	2.0999998E 1	8	3.3679997E 3	13	8.3999997E 1
3	1.1900000E 2	9	2.9119997E 3	14	1.5000000E 1
4	3.9199997E 2	10	1.9669998E 3	15	1.0000000E 0
5	1.0360000E 3				
N = 31					
0	0. E 0	11	3.0817054E 7	22	1.4044872E 7
2	4.4999999E 1	13	8.7527003E 7	24	1.9900398E 6
1	0. E 0	12	5.5619190E 7	23	5.7343643E 6
3	5.7499997E 2	14	1.2056762E 8	25	5.7948793E 5
4	4.7599996E 3	15	1.4573224E 8	26	1.3908298E 5
5	3.0827997E 4	16	1.5480789E 8	27	2.6704997E 4
6	1.5679297E 5	17	1.4461485E 8	28	3.9199998E 3
7	6.3953493E 5	18	1.1872603E 8	29	4.1999998E 2
8	2.1543597E 6	19	8.5501311E 7	30	3.0999998E 1
9	6.1151993E 6	20	5.3855249E 7	31	1.0000000E 0
10	1.4800927E 7	21	2.9551231E 7		
N = 63					
0	0. E 0	22	1.8661421E 16	43	9.1295317E 15
1	0. E 0	23	3.4711347E 16	44	4.2440660E 15
2	9.3000000E 1	24	6.0261134E 16	45	1.8317456E 15
3	2.5109998E 3	25	9.7765238E 16	46	7.3234654E 14
4	4.5879995E 4	26	1.4837992E 17	47	2.7052537E 14
5	6.4938793E 5	27	2.1086189E 17	48	9.2053133E 13
6	7.3313442E 6	28	2.8078501E 17	49	2.8754434E 13
7	6.8206223E 7	29	3.5055742E 17	50	8.2122160E 12
8	5.3693306E 8	30	4.1053458E 17	51	2.1343423E 12
9	3.6451927E 9	31	4.5111079E 17	52	5.0200919E 11
10	2.1649648E 10	32	4.6520090E 17	53	1.0615585E 11
11	1.1378094E 11	33	4.5024307E 17	54	2.0022493E 10
12	5.3408157E 11	34	4.0895226E 17	55	3.3359610E 9
13	2.2562160E 12	35	3.4852302E 17	56	4.8506437E 8
14	8.6328226E 12	36	2.7859991E 17	57	6.0614169E 7
15	3.0078571E 13	37	2.0879491E 17	58	6.3794584E 6
16	9.5869747E 13	38	1.4661755E 17	59	5.4978495E 5
17	2.8062814E 14	39	9.6394563E 16	60	3.7199997E 4
18	7.5696746E 14	40	5.9282036E 16	61	1.8599999E 3
19	1.8870963E 15	41	3.4066575E 16	62	6.3000000E 1
20	4.3590252E 15	42	1.8269100E 16	63	1.0000000E 0
21	9.3503265E 15				

I	A(I)	I	A(I)	I	A(I)
N = 127					
0	0. E 0	43	5.1322422E 33	86	2.5068906E 33
1	0. E 0	44	1.0020557E 34	87	1.1949864E 33
2	1.8900000E 2	45	1.8893005E 34	88	5.4934755E 32
3	1.0478999E 4	46	3.4410860E 34	89	2.4342988E 32
4	4.0101596E 5	47	6.0565392E 34	90	1.0392342E 32
5	1.1835178E 7	48	1.0304493E 35	91	4.2716872E 31
6	2.8079643E 8	49	1.6952360E 35	92	1.6897762E 31
7	5.5463721E 9	50	2.6974512E 35	93	6.4277472E 30
8	9.3583381E 10	51	4.1524651E 35	94	2.3496611E 30
9	1.3747458E 12	52	6.1856810E 35	95	8.2478876E 29
10	1.7842929E 13	53	8.9184543E 35	96	2.7779318E 29
11	2.0702452E 14	54	1.2447877E 36	97	8.9694439E 28
12	2.1678964E 15	55	1.6822080E 36	98	2.7737637E 28
13	2.0651840E 16	56	2.2014555E 36	99	8.2072324E 27
14	1.8016995E 17	57	2.7902652E 36	100	2.3210036E 27
15	1.4477179E 18	58	3.4256150E 36	101	6.2660908E 26
16	1.0767087E 19	59	4.0741206E 36	102	1.6128968E 26
17	7.4436275E 19	60	4.6942821E 36	103	3.9528030E 25
18	4.8014877E 20	61	5.2405285E 36	104	9.2095567E 24
19	2.8994527E 21	62	5.6685934E 36	105	2.0365427E 24
20	1.6439586E 22	63	5.9413712E 36	106	4.2666591E 23
21	8.7750841E 22	64	6.0341936E 36	107	8.4520714E 22
22	4.4201073E 23	65	5.9384926E 36	108	1.5796899E 22
23	2.1055785E 24	66	5.6630976E 36	109	2.7788484E 21
24	9.5042216E 24	67	5.2329003E 36	110	4.5885419E 20
25	4.0723190E 25	68	4.6851578E 36	111	7.0908030E 19
26	1.6590295E 26	69	4.0642031E 36	112	1.0220155E 19
27	6.4357836E 26	70	3.4155840E 36	113	1.3685629E 18
28	2.3805598E 27	71	2.7807052E 36	114	1.6955570E 17
29	8.4068946E 27	72	2.1928059E 36	115	1.9333826E 16
30	2.8377697E 28	73	1.6747433E 36	116	2.0172916E 15
31	9.1658343E 28	74	1.2386232E 36	117	1.9128082E 14
32	2.8356588E 29	75	8.8696313E 35	118	1.6347606E 13
33	8.4105718E 29	76	6.1485381E 35	119	1.2467625E 12
34	2.3936507E 30	77	4.1252912E 35	120	8.3810028E 10
35	6.5419528E 30	78	2.6783169E 35	121	4.8885821E 9
36	1.7182619E 31	79	1.6822607E 35	122	2.4239656E 8
37	4.3401871E 31	80	1.0219721E 35	123	9.9336083E 6
38	1.0549844E 32	81	6.0031642E 34	124	3.2289597E 5
39	2.4692474E 32	82	3.4086933E 34	125	7.8119997E 3
40	5.5681260E 32	83	1.8703507E 34	126	1.2700000E 2
41	1.2103413E 33	84	9.9136964E 33	127	1.0000000E 0
42	2.5373143E 33	85	5.0741606E 33		

I	A(I)	I	A(I)	I	A(I)
N = 255					
0	0. E 0	10	1.1471767E 16	20	2.1128190E 28
1	0. E 0	11	2.7872717E 17	21	2.4769118E 29
2	3.8099999E 2	12	6.1395853E 18	22	2.7542641E 30
3	4.2798996E 4	13	1.2358827E 20	23	2.9114812E 31
4	3.3497516E 6	14	2.2888621E 21	24	2.9316804E 32
5	2.0176387E 8	15	3.9225366E 22	25	2.8172105E 33
6	9.8078245E 9	16	6.2514522E 23	26	2.5879850E 34
7	3.9869002E 11	17	9.3056751E 24	27	2.2762789E 35
8	1.3903531E 13	18	1.2987585E 26	28	1.9197300E 36
9	4.2395332E 14	19	1.7052775E 27	29	1.5544948E 37

I	A(I)	I	A(I)	I	A(I)
30	1.2100827E 38	87	2.1549805E 69	144	1.9321076E 74
31	9.0661405E 38	88	4.1608024E 69	145	1.4892618E 74
32	6.5445957E 39	89	7.8950674E 69	146	1.1297317E 74
33	4.5565611E 40	90	1.4723806E 70	147	8.4339062E 73
34	3.0626610E 41	91	2.6990372E 70	148	6.1960555E 73
35	1.9890983E 42	92	4.8636219E 70	149	4.4793781E 73
36	1.2493220E 43	93	8.6160688E 70	150	3.1865295E 73
37	7.5944690E 43	94	1.5006931E 71	151	2.2304724E 73
38	4.4714683E 44	95	2.5700502E 71	152	1.5361527E 73
39	2.5517584E 45	96	4.3280337E 71	153	1.0409010E 73
40	1.4123949E 46	97	7.1675411E 71	154	6.9390461E 72
41	7.5870889E 46	98	1.1673743E 72	155	4.5507427E 72
42	3.9578368E 47	99	1.8699897E 72	156	2.9358420E 72
43	2.0060988E 48	100	2.9463547E 72	157	1.8630546E 72
44	9.8854057E 48	101	4.5664002E 72	158	1.1628791E 72
45	4.7381511E 49	102	6.9619583E 72	159	7.1389108E 71
46	2.2100882E 50	103	1.0441949E 73	160	4.3101168E 71
47	1.0036924E 51	104	1.5408048E 73	161	2.5590327E 71
48	4.4399375E 51	105	2.2369263E 73	162	1.4940364E 71
49	1.9139225E 52	106	3.1953234E 73	163	8.5765524E 70
50	8.0430552E 52	107	4.4911457E 73	164	4.8405729E 70
51	3.2963797E 53	108	6.2115171E 73	165	2.6858286E 70
52	1.3180623E 54	109	8.4538506E 73	166	1.4649441E 70
53	5.1436738E 54	110	1.1322567E 74	167	7.8539355E 69
54	1.9597441E 55	111	1.4923985E 74	168	4.1384535E 69
55	7.2921824E 55	112	1.9359297E 74	169	2.1430524E 69
56	2.6508538E 56	113	2.4715729E 74	170	1.0905095E 69
57	9.4170877E 56	114	3.1056304E 74	171	5.4523603E 68
58	3.2702226E 57	115	3.8408828E 74	172	2.6782516E 68
59	1.1104276E 58	116	4.6755085E 74	173	1.2923736E 68
60	3.6878497E 58	117	5.6021379E 74	174	6.1254986E 67
61	1.1982278E 59	118	6.6071735E 74	175	2.8514318E 67
62	3.8097619E 59	119	7.6704965E 74	176	1.3034694E 67
63	1.1856423E 60	120	8.7656705E 74	177	5.8506129E 66
64	3.6125006E 60	121	9.8607030E 74	178	2.5781546E 65
65	1.0778494E 61	122	1.0919380E 75	179	1.1152341E 66
66	3.1499109E 61	123	1.1903124E 75	180	4.7349257E 65
67	9.0181818E 61	124	1.2773250E 75	181	1.9728253E 65
68	2.5299256E 62	125	1.3493454E 75	182	8.0654503E 64
69	6.9558280E 62	126	1.4032305E 75	183	3.2349463E 64
70	1.8746651E 63	127	1.4365513E 75	184	1.2727280E 64
71	4.9534854E 63	128	1.4477740E 75	185	4.9109261E 63
72	1.2834716E 64	129	1.4363766E 75	186	1.8581343E 63
73	3.2615423E 64	130	1.4028891E 75	187	6.8928814E 62
74	8.1300257E 64	131	1.3488530E 75	188	2.5064312E 62
75	1.9882054E 65	132	1.2767033E 75	189	8.9322430E 61
76	4.7708636E 65	133	1.1895878E 75	190	3.1191093E 61
77	1.1234736E 66	134	1.0911400E 75	191	1.0670343E 61
78	2.5966922E 66	135	9.8522884E 74	192	3.5753057E 60
79	5.8915462E 66	136	8.7571144E 74	193	1.1731155E 60
80	1.3123411E 67	137	7.6620655E 74	194	3.7684564E 59
81	2.8703068E 67	138	6.5990953E 74	195	1.1848957E 59
82	6.1649237E 67	139	5.5945947E 74	196	3.6457379E 58
83	1.3004590E 68	140	4.6686313E 74	197	1.0974130E 58
84	2.6945435E 68	141	3.8347535E 74	198	3.2308798E 57
85	5.4845586E 68	142	3.1002846E 74	199	9.3007822E 56
86	1.0967628E 69	143	2.4670068E 74	200	2.6172398E 56

I	A(I)	I	A(I)	I	A(I)
201	7.1972309E 55	220	1.2209736E 43	238	1.2353370E 26
202	1.9335369E 55	221	1.9424180E 42	239	8.8236809E 24
203	5.0730177E 54	222	2.9882747E 41	240	5.9069637E 23
204	1.2994608E 54	223	4.4419402E 40	241	3.6917886E 22
205	3.2485746E 53	224	6.3739573E 39	242	2.1445708E 21
206	7.9231659E 52	225	8.8209268E 38	243	1.1520226E 20
207	1.8845924E 52	226	1.1761004E 38	244	5.6889046E 18
208	4.3699679E 51	227	1.5091260E 37	245	2.5646273E 17
209	9.8742203E 50	228	1.8614412E 36	246	1.0467692E 16
210	2.1732238E 50	229	2.2042961E 35	247	3.8295809E 14
211	4.6568034E 49	230	2.5026468E 34	248	1.2403298E 13
212	9.7106528E 48	231	2.7202172E 33	249	3.3008743E 11
213	1.9695700E 48	232	2.8261470E 32	250	8.4357226E 9
214	3.8835740E 47	233	2.8017319E 31	251	1.6871173E 8
215	7.4403308E 46	234	2.6453637E 30	252	2.6883357E 6
216	1.3842178E 46	235	2.3740005E 29	253	3.2003999E 4
217	2.4992289E 45	236	2.0203896E 28	254	2.5500000E 2
218	4.3764374E 44	237	1.6265558E 27	255	1.0000000E 0
219	7.4277433E 43				

N = 511

0	0. E 0	36	1.7475956E 54	72	1.2334938E 88
1	0. E 0	37	2.3041678E 55	73	7.5194716E 88
2	7.6499999E 2	38	2.9497778E 56	74	4.5108573E 89
3	1.7297498E 5	39	3.6692783E 57	75	2.6633701E 90
4	2.7376117E 7	40	4.4379869E 58	76	1.5480373E 91
5	3.3309134E 9	41	5.2226148E 59	77	8.8589792E 91
6	3.2771594E 11	42	5.9835006E 60	78	4.9924204E 92
7	2.7018898E 13	43	6.6779550E 61	79	2.7709884E 93
8	1.9149117E 15	44	7.2643386E 62	80	1.5150374E 94
9	1.1891106E 17	45	7.7062898E 63	81	8.1610177E 94
10	6.5661520E 18	46	7.9765126E 64	82	4.3317464E 95
11	3.2624064E 20	47	8.0595562E 65	83	2.2659132E 96
12	1.4725954E 22	48	7.9532081E 66	84	1.1682809E 97
13	6.0872431E 23	49	7.6693305E 67	85	5.9379378E 97
14	2.3199633E 25	50	7.2272424E 68	86	2.9755514E 98
15	8.1991988E 26	51	6.6609505E 69	87	1.4702813E 99
16	2.7005916E 28	52	6.0056900E 70	88	7.1645817E 99
17	8.3259917E 29	53	5.2992852E 71	89	3.4434485E100
18	2.4119546E 31	54	4.5778088E 72	90	1.6325342E101
19	6.5877420E 32	55	3.8728994E 73	91	7.6357063E101
20	1.7016057E 34	56	3.2099597E 74	92	3.5237559E102
21	4.1679517E 35	57	2.6072878E 75	93	1.6046550E103
22	9.7050896E 36	58	2.0760634E 76	94	7.2115027E103
23	2.1530913E 38	59	1.6210104E 77	95	3.1987901E104
24	4.5603517E 39	60	1.2415131E 78	96	1.4005810E105
25	9.2388797E 40	61	9.3295275E 78	97	6.0539503E105
26	1.7933762E 42	62	6.8806444E 79	98	2.5835814E106
27	3.3407388E 43	63	4.9816602E 80	99	1.0886836E107
28	5.9809307E 44	64	3.5416474E 81	100	4.5302296E107
29	1.0304814E 46	65	2.4730327E 82	101	1.8617414E108
30	1.7108245E 47	66	1.6964906E 83	102	7.5568364E108
31	2.7401621E 48	67	1.1435905E 84	103	3.0298569E109
32	4.2386803E 49	68	7.5767789E 84	104	1.2000651E110
33	6.3389373E 50	69	4.9350097E 85	105	4.6959819E110
34	9.1738966E 51	70	3.1606207E 86	106	1.8156178E111
35	1.2859911E 53	71	1.9907952E 87	107	6.9364229E111

I	A(I)	I	A(I)	I	A(I)
108	2.6187607E112	165	4.8359080E137	222	1.2667989E150
109	9.7710332E112	166	1.0140384E138	223	1.6490878E150
110	3.6033302E113	167	2.1074136E138	224	2.1297211E150
111	1.3134707E114	168	4.3408656E138	225	2.7286512E150
112	4.7328501E114	169	8.8622897E138	226	3.4683508E150
113	1.6859455E115	170	1.7933715E139	227	4.3737200E150
114	5.9376512E115	171	3.5971690E139	228	5.4718615E150
115	2.0676042E116	172	7.1520236E139	229	6.7916983E150
116	7.1192202E116	173	1.4095668E140	230	8.3634173E150
117	2.4240396E117	174	2.7538601E140	231	1.0217723E151
118	8.1624180E117	175	5.3334510E140	232	1.2384887E151
119	2.7183078E118	176	1.0239894E141	233	1.4893619E151
120	8.9538025E118	177	1.9490084E141	234	1.7769744E151
121	2.9172478E119	178	3.6776735E141	235	2.1034742E151
122	9.4020658E119	179	6.8799270E141	236	2.4704189E151
123	2.9976739E120	180	1.2760140E142	237	2.8786147E151
124	9.4554609E120	181	2.3463762E142	238	3.3279597E151
125	2.9508296E121	182	4.2777941E142	239	3.8172987E151
126	9.1115866E121	183	7.7327040E142	240	4.3442978E151
127	2.7839229E122	184	1.3859299E143	241	4.9053515E151
128	8.4170158E122	185	2.4629663E143	242	5.4955279E151
129	2.5183773E123	186	4.3400204E143	243	6.1085615E151
130	7.4570780E123	187	7.5831517E143	244	6.7369009E151
131	2.1853698E124	188	1.3138349E144	245	7.3718136E151
132	6.3388764E124	189	2.2572170E144	246	8.0035505E151
133	1.8199226E125	190	3.8455221E144	247	8.6215694E151
134	5.1721233E125	191	6.4967299E144	248	9.2148064E151
135	1.4550621E126	192	1.0884278E145	249	9.7719957E151
136	4.0523979E126	193	1.8083288E145	250	1.0282015E152
137	1.1173293E127	194	2.9794468E145	251	1.0734253E152
138	3.0500673E127	195	4.8683492E145	252	1.1118971E152
139	8.2435949E127	196	7.8890163E145	253	1.1427652E152
140	2.2060866E128	197	1.2678448E146	254	1.1653314E152
141	5.8458350E128	198	2.0207772E146	255	1.1790770E152
142	1.5339372E129	199	3.1943800E146	256	1.1836828E152
143	3.9858808E129	200	5.0081488E146	257	1.1790411E152
144	1.0256875E130	201	7.7874783E146	258	1.1652604E152
145	2.6139542E130	202	1.2010243E147	259	1.1426608E152
146	6.5976707E130	203	1.8371658E147	260	1.1117616E152
147	1.6493411E131	204	2.7873566E147	261	1.0732618E152
148	4.0838960E131	205	4.1945986E147	262	1.0280135E152
149	1.0016130E132	206	6.2610579E147	263	9.7699111E151
150	2.4333407E132	207	9.2697959E147	264	9.2125593E151
151	5.8559826E132	208	1.3613298E148	265	8.6192034E151
152	1.3960678E133	209	1.9830454E148	266	8.0011096E151
153	3.2971504E133	210	2.8653881E148	267	7.3693395E151
154	7.7145751E133	211	4.1069639E148	268	6.7344336E151
155	1.7883042E134	212	5.8391550E148	269	6.1061372E151
156	4.1071616E134	213	8.2352298E148	270	5.4931781E151
157	9.3460455E134	214	1.1521336E149	271	4.9031032E151
158	2.1072403E135	215	1.5989546E149	272	4.3421729E151
159	4.7077615E135	216	2.2013042E149	273	3.8153139E151
160	1.0421806E136	217	3.0063468E149	274	3.3261267E151
161	2.2861953E136	218	4.0730291E149	275	2.8769402E151
162	4.9698001E136	219	5.4741867E149	276	2.4689053E151
163	1.0706141E137	220	7.2987643E149	277	2.1021202E151
164	2.2856430E137	221	9.6540754E149	278	1.7757753E151

I	A(I)	I	A(I)	I	A(I)
279	1.4883105E151	336	1.0212291E141	393	2.7044836E118
280	1.2375757E151	337	5.3188553E140	394	8.1142755E117
281	1.0209870E151	338	2.7462097E140	395	2.4095537E117
282	8.3567282E150	339	1.4055922E140	396	7.0761126E116
283	6.7860528E150	340	7.1315565E139	397	2.0549186E116
284	5.4671408E150	341	3.5867225E139	398	5.9007369E115
285	4.3698087E150	342	1.7880868E139	399	1.6753243E115
286	3.4651392E150	343	8.8357925E138	400	4.7026354E114
287	2.7260379E150	344	4.3276985E138	401	1.3049731E114
288	2.1276136E150	345	2.1009290E138	402	3.5797048E113
289	1.6474032E150	346	1.0108733E138	403	9.7061051E112
290	1.2654643E150	347	4.8205986E137	404	2.6011238E112
291	9.6435955E149	348	2.2783045E137	405	6.8890733E111
292	7.2906052E149	349	1.0671282E137	406	1.8030552E111
293	5.4678896E149	350	4.9533910E136	407	4.6630456E110
294	4.0682120E149	351	2.2785413E136	408	1.1915327E110
295	3.0026934E149	352	1.0386429E136	409	3.0080181E109
296	2.1985572E149	353	4.6915595E135	410	7.5016145E108
297	1.5969068E149	354	2.0998881E135	411	1.8479477E108
298	1.1506202E149	355	9.3129894E134	412	4.4961967E107
299	8.2241398E148	356	4.0924362E134	413	1.0803905E107
300	5.8310979E148	357	1.7818052E134	414	2.5636235E106
301	4.1011602E148	358	7.6861584E133	415	6.0065224E105
302	2.8612432E148	359	3.2848411E133	416	1.3894525E105
303	1.9801102E148	360	1.3907854E133	417	3.1730105E104
304	1.3592689E148	361	5.8335277E132	418	7.1525490E103
305	9.2554485E147	362	2.4238849E132	419	1.5913475E103
306	6.2511541E147	363	9.9766884E131	420	3.4941083E102
307	4.1878202E147	364	4.0675998E131	421	7.5705254E101
308	2.7827566E147	365	1.6426721E131	422	1.6183929E101
309	1.8340704E147	366	6.5706400E130	423	3.4131794E100
310	1.1989591E147	367	2.6031030E130	424	7.1006651E 99
311	7.7738159E146	368	1.0213734E130	425	1.4569680E 99
312	4.9991872E146	369	3.9688950E129	426	2.9482014E 98
313	3.1885515E146	370	1.5273142E129	427	5.8825292E 97
314	2.0170185E146	371	5.8202634E128	428	1.15[illegible]2125E 97
315	1.2654415E146	372	2.1963100E128	429	2.2441150E 96
316	7.8737795E145	373	8.2065831E127	430	4.2894274E 95
317	4.8587712E145	374	3.0361939E127	431	8.0800407E 94
318	2.9734772E145	375	1.1121808E127	432	1.4997675E 94
319	1.8046398E145	376	4.0334806E126	433	2.7426157E 93
320	1.0861676E145	377	1.4481809E126	434	4.9404826E 92
321	6.4829998E144	378	5.1473445E125	435	8.7653260E 91
322	3.8372526E144	379	1.8110901E125	436	1.5314052E 91
323	2.2522787E144	380	6.3077109E124	437	2.6342842E 90
324	1.3109113E144	381	2.1744853E124	438	4.4607775E 89
325	7.5659910E143	382	7.4194530E123	439	7.4345906E 88
326	4.3300340E143	383	2.5055049E123	440	1.2193342E 88
327	2.4572049E143	384	8.3734325E122	441	1.9675520E 87
328	1.3826345E143	385	2.7693195E122	442	3.1230824E 86
329	7.7140180E142	386	9.0631672E121	443	4.8753754E 85
330	4.2672899E142	387	2.9349439E121	444	7.4836084E 84
331	2.3405225E142	388	9.4038918E120	445	1.1292776E 84
332	1.2727801E142	389	2.9811110E120	446	1.6748752E 83
333	6.8622165E141	390	9.3494366E119	447	2.4409494E 82
334	3.6680588E141	391	2.9007035E119	448	3.4948542E 81
335	1.9438342E141	392	8.9023570E118	449	4.9146144E 80

I	A(I)	I	A(I)	I	A(I)
450	6.7862936E 79	471	5.1088740E 59	492	1.6237257E 34
451	9.1991532E 78	472	4.3387268E 58	493	6.2704596E 32
452	1.2238281E 78	473	3.5849489E 57	494	2.2894079E 31
453	1.5974670E 77	474	2.8800732E 56	495	7.8784955E 29
454	2.0453120E 76	475	2.2481484E 55	496	2.5465740E 28
455	2.5678903E 75	476	1.7038523E 54	497	7.7013018E 26
456	3.1604653E 74	477	1.2528271E 53	498	2.1693719E 25
457	3.8119467E 73	478	8.9299655E 51	499	5.6629968E 23
458	4.5042481E 72	479	6.1650122E 50	500	1.3618374E 22
459	5.2123148E 71	480	4.1185706E 49	501	2.9960305E 20
460	5.9049912E 70	481	2.6598987E 48	502	5.9800774E 18
461	6.5468072E 69	482	1.6589734E 47	503	1.0721211E 17
462	7.1006257E 68	483	9.9813323E 45	504	1.7051562E 15
463	7.5309313E 67	484	5.7862550E 44	505	2.3682633E 13
464	7.8074092E 66	485	3.2278560E 43	506	2.8137678E 11
465	7.9083306E 65	486	1.7303896E 42	507	2.7803922E 9
466	7.8232586E 64	487	8.9011443E 40	508	2.1935438E 7
467	7.5546145E 63	488	4.3865824E 39	509	1.2953999E 5
468	7.1178055E 62	489	2.0674381E 38	510	5.1099998E 2
469	6.5398343E 61	490	9.3013185E 36	511	1.0000000E 0
470	5.8565418E 60	491	3.9862629E 35		

ERROR RATE EQUATION COEFFICIENTS

HAMMING SINGLE ERROR CORRECTING /DOUBLE ERROR DETECTING CODES

DETECTION WITHOUT CORRECTION OCCURS IF CHECK WORD IS NON-ZERO AND OVERALL PARITY CHECK IS SATISFIED.

I	A(I)	B(I)	I	A(I)	B(I)
N = 8					
0	0. E 0	0. E 0	5	2.7999999E 1	0. E 0
1	0. E 0	0. E 0	6	0. E 0	2.7999999E 1
2	0. E 0	2.7999999E 1	7	7.9999999E 0	0. E 0
3	2.7999999E 1	0. E 0	8	1.0000000E 0	0. E 0
4	6.9999999E 0	5.5999998E 1			
N = 16					
0	0. E 0	0. E 0	9	6.2799993E 3	0. E 0
1	0. E 0	0. E 0	10	2.7999997E 2	7.5599993E 3
2	0. E 0	1.2000000E 2	11	2.9399997E 3	0. E 0
3	1.3999999E 2	0. E 0	12	1.0500000E 2	1.6799998E 3
4	3.4999998E 1	1.6799998E 3	13	4.1999998E 2	0. E 0
5	1.4279999E 3	0. E 0	14	0. E 0	1.2000000E 2
6	1.6799999E 2	7.5599995E 3	15	1.6000000E 1	0. E 0
7	5.1599994E 3	0. E 0	16	1.0000000E 0	0. E 0
8	4.3499996E 2	1.1999999E 4			
N = 32					
0	0. E 0	0. E 0	17	2.9942274E 8	0. E 0
1	0. E 0	0. E 0	18	8.2807187E 6	4.5671424E 8
2	0. E 0	4.9599999E 2	19	2.0422734E 8	0. E 0
3	6.1999997E 2	0. E 0	20	4.4148644E 6	2.1872902E 8
4	1.5499999E 2	3.4719996E 4	21	8.3406479E 7	0. E 0
5	3.5587997E 4	0. E 0	22	1.3830958E 6	6.2500456E 7
6	5.2079994E 3	8.7841592E 5	23	1.9779237E 7	0. E 0
7	7.9632791E 5	0. E 0	24	2.4784497E 5	1.0187838E 7
8	8.2614992E 4	1.0187838E 7	25	2.5695277E 6	0. E 0
9	8.2695589E 6	0. E 0	26	2.2567998E 4	8.7841591E 5
10	6.2867992E 5	6.2500456E 7	27	1.6578798E 5	0. E 0
11	4.5617980E 7	0. E 0	28	1.0850000E 3	3.4719996E 4
12	2.6489185E 6	2.1872902E 8	29	4.3399997E 3	0. E 0
13	1.4314619E 8	0. E 0	30	0. E 0	4.9599999E 2
14	6.4405589E 6	4.5671424E 8	31	3.1999999E 1	0. E 0
15	2.5629986E 8	0. E 0	32	1.0000000E 0	0. E 0
16	9.3981135E 6	5.8228405E 8			
N = 64					
0	0. E 0	0. E 0	9	4.1821257E 9	0. E 0
1	0. E 0	0. E 0	10	3.6977662E 8	1.4910661E 11
2	0. E 0	2.0159999E 3	11	1.3543059E 11	0. E 0
3	2.6039998E 3	0. E 0	12	9.6218882E 9	3.2328973E 12
4	6.5099998E 2	6.2495995E 5	13	2.7902975E 12	0. E 0
5	6.9526793E 5	0. E 0	14	1.6356852E 11	4.7107946E 13
6	1.0936798E 5	7.3807768E 7	15	3.8711394E 13	0. E 0
7	7.5537566E 7	0. E 0	16	1.9083105E 12	4.8089357E 14
8	8.6492779E 6	4.3569705E 9	17	3.7649789E 14	0. E 0

I	A(I)	B(I)	I	A(I)	B(I)
18	1.5827722E 13	3.5454114E 15	42	8.2387484E 14	7.9091995E 16
19	2.6440638E 15	0. E 0	43	2.7398631E 16	0. E 0
20	9.3799435E 13	1.9313161E 16	44	2.1075875E 14	1.9313161E 16
21	1.3709352E 16	0. E 0	45	6.0758117E 15	0. E 0
22	4.3155349E 14	7.9091995E 16	46	4.0448622E 13	3.5454114E 15
23	5.3372767E 16	0. E 0	47	1.0028719E 15	0. E 0
24	1.4686466E 15	2.4673263E 17	48	5.7249314E 12	4.8089357E 14
25	1.5802637E 17	0. E 0	49	1.2080757E 14	0. E 0
26	3.8184809E 15	5.9215830E 17	50	5.8417331E 11	4.7107946E 13
27	3.5924182E 17	0. E 0	51	1.0346558E 13	0. E 0
28	7.6478410E 15	1.1012890E 18	52	4.1694847E 10	3.2328973E 12
29	6.3134243E 17	0. E 0	53	6.0816503E 11	0. E 0
30	1.1867338E 16	1.5949704E 18	54	1.9967937E 9	1.4910661E 11
31	8.6164534E 17	0. E 0	55	2.3358453E 10	0. E 0
32	1.4317370E 16	1.8039887E 18	56	6.0544945E 7	4.3569705E 9
33	9.1544397E 17	0. E 0	57	5.4567853E 8	0. E 0
34	1.3449650E 16	1.5949704E 18	58	1.0572238E 6	7.3807767E 7
35	7.5747529E 17	0. E 0	59	6.9292433E 6	0. E 0
36	9.8329383E 15	1.1012890E 18	60	9.7649993E 3	6.2495995E 5
37	4.8739483E 17	0. E 0	61	3.9059997E 4	0. E 0
38	5.5808567E 15	5.9215830E 17	62	0. E 0	2.0159999E 3
39	2.4301206E 17	0. E 0	63	6.3999998E 1	0. E 0
40	2.4477445E 15	2.4673263E 17	64	1.0000000E 0	0. E 0
41	9.3348610E 16	0. E 0			

N = 128

0	0. E 0	0. E 0	31	1.2003604E 29	0. E 0
1	0. E 0	0. E 0	32	2.8863401E 27	1.4662611E 30
2	0. E 0	8.1279998E 3	33	1.1246232E 30	0. E 0
3	1.0667998E 4	0. E 0	34	2.4927482E 28	1.1918272E 31
4	2.6669998E 3	1.0582655E 7	35	8.9356035E 30	0. E 0
5	1.2236194E 7	0. E 0	36	1.8312272E 29	8.2690103E 31
6	1.9842478E 6	5.3812798E 9	37	6.0584490E 31	0. E 0
7	5.8271685E 9	0. E 0	38	1.1509788E 30	4.9237662E 32
8	6.9813629E 8	1.4185322E 12	39	3.5242318E 32	0. E 0
9	1.4683292E 12	0. E 0	40	6.2208773E 30	2.5281645E 33
10	1.3845528E 11	2.2507387E 14	41	1.7671539E 33	0. E 0
11	2.2486746E 14	0. E 0	42	2.9040875E 31	1.1240201E 34
12	1.7377477E 13	2.3540680E 16	43	7.6695566E 33	0. E 0
13	2.2819737E 16	0. E 0	44	1.1754639E 32	4.3428050E 34
14	1.4859963E 15	1.7254542E 18	45	2.8913562E 34	0. E 0
15	1.3278879E 18	0. E 0	46	4.1390579E 32	1.4627070E 35
16	9.1155268E 16	9.2613750E 19	47	9.4976252E 34	0. E 0
17	8.5203363E 19	0. E 0	48	1.2715828E 33	4.3064273E 35
18	4.1663318E 18	3.7626604E 21	49	2.7256853E 35	0. E 0
19	3.3796015E 21	0. E 0	50	3.4168382E 33	1.1108824E 36
20	1.4606526E 20	1.1872184E 23	51	6.8499163E 35	0. E 0
21	1.0419043E 23	0. E 0	52	8.0476589E 33	2.5158220E 36
22	4.0188810E 21	2.9695877E 24	53	1.5104135E 36	0. E 0
23	2.5475892E 24	0. E 0	54	1.6644285E 34	5.0105467E 36
24	8.8399496E 22	5.9875923E 25	55	2.9269957E 36	0. E 0
25	5.0227412E 25	0. E 0	56	3.0273547E 34	8.7879781E 36
26	1.5782256E 24	9.8675520E 26	57	4.9917207E 36	0. E 0
27	8.0948132E 26	0. E 0	58	4.8483197E 34	1.3588670E 37
28	2.3160797E 25	1.3446496E 28	59	7.4997357E 36	0. E 0
29	1.0787453E 28	0. E 0	60	6.8431865E 34	1.8540473E 37
30	2.8237916E 26	1.5301185E 29	61	9.9348105E 36	0. E 0

I	A(I)	B(I)	I	A(I)	B(I)
62	8.5184579E 34	2.2334847E 37	96	8.6590204E 27	1.4662611E 30
63	1.1609964E 37	0. E 0	97	3.6748762E 29	0. E 0
64	9.3559102E 34	2.3764012E 37	98	9.2243860E 26	1.5301185E 29
65	1.1972686E 37	0. E 0	99	3.5944870E 28	0. E 0
66	9.0680360E 34	2.2334847E 37	100	8.2717134E 25	1.3446496E 28
67	1.0895997E 37	0. E 0	101	2.9476126E 27	0. E 0
68	7.7556114E 34	1.8540473E 37	102	6.1915003E 24	9.8675520E 26
69	8.7493608E 36	0. E 0	103	2.0081771E 26	0. E 0
70	5.8514204E 34	1.3588670E 37	104	3.8306447E 23	5.9875923E 25
71	6.1962892E 36	0. E 0	105	1.1246100E 25	0. E 0
72	3.8923132E 34	8.7879781E 36	106	1.9363699E 22	2.9695876E 24
73	3.8675492E 36	0. E 0	107	5.1118663E 23	0. E 0
74	2.2808837E 34	5.0105467E 36	108	7.8875237E 20	1.1872184E 23
75	2.1255863E 36	0. E 0	109	1.8575748E 22	0. E 0
76	1.1761962E 34	2.5158220E 36	110	2.5460916E 19	3.7626604E 21
77	1.0273829E 36	0. E 0	111	5.2976223E 20	0. E 0
78	5.3302675E 33	1.1108824E 36	112	6.3808686E 17	9.2613750E 19
79	4.3605776E 35	0. E 0	113	1.1588818E 19	0. E 0
80	2.1193047E 33	4.3064273E 35	114	1.2100256E 16	1.7254542E 18
81	1.6222885E 35	0. E 0	115	1.8888952E 17	0. E 0
82	7.3783206E 32	1.4627070E 35	116	1.6798228E 14	2.3540680E 16
83	5.2790442E 34	0. E 0	117	2.2085725E 15	0. E 0
84	2.2440675E 32	4.3428050E 34	118	1.6337723E 12	2.2507387E 14
85	1.4987856E 34	0. E 0	119	1.7594368E 13	0. E 0
86	5.9464650E 31	1.1240201E 34	120	1.0472044E 10	1.4185322E 12
87	3.7018770E 33	0. E 0	121	8.8698611E 10	0. E 0
88	1.3685930E 31	2.5281645E 33	122	4.0346372E 7	5.3812796E 9
89	7.9277742E 32	0. E 0	123	2.5233017E 8	0. E 0
90	2.7260025E 30	4.9237662E 32	124	8.2676995E 4	1.0582655E 7
91	1.4664229E 32	0. E 0	125	3.3070797E 5	0. E 0
92	4.6798030E 29	8.2690103E 31	126	0. E 0	8.1279998E 3
93	2.3325509E 31	0. E 0	127	1.2800000E 2	0. E 0
94	6.8917156E 28	1.1918272E 31	128	1.0000000E 0	0. E 0
95	3.1744498E 30	0. E 0			

N = 256

0	0. E 0	0. E 0	21	2.6881936E 29	0. E 0
1	0. E 0	0. E 0	22	1.1300992E 28	3.3533137E 31
2	0. E 0	3.2639999E 4	23	3.1869075E 31	0. E 0
3	4.3179997E 4	0. E 0	24	1.2176934E 30	3.3121260E 33
4	1.0794999E 4	1.7410174E 8	25	3.1103785E 33	0. E 0
5	2.0511360E 8	0. E 0	26	1.0876437E 32	2.7308223E 35
6	3.3732213E 7	3.6709350E 11	27	2.5350774E 35	0. E 0
7	4.0849783E 11	0. E 0	28	8.1604264E 33	1.9025451E 37
8	5.0008099E 10	4.0806335E 14	29	1.7464678E 37	0. E 0
9	4.3785686E 14	0. E 0	30	5.2013671E 35	1.1318174E 39
10	4.2545484E 13	2.7773698E 17	31	1.0276223E 39	0. E 0
11	2.9019893E 17	0. E 0	32	2.8439732E 37	5.8017053E 40
12	2.3311063E 16	1.2681218E 20	33	5.2110206E 40	0. E 0
13	1.2972785E 20	0. E 0	34	1.3452854E 39	2.5829480E 42
14	8.8599968E 18	4.1312897E 22	35	2.2953644E 42	0. E 0
15	4.1514229E 22	0. E 0	36	5.5464196E 40	1.0057507E 44
16	2.4606318E 21	1.0039377E 25	37	8.8437908E 43	0. E 0
17	9.9308201E 24	0. E 0	38	2.0062049E 42	3.4464487E 45
18	5.1890383E 23	1.8818911E 27	39	2.9989053E 45	0. E 0
19	1.8351534E 27	0. E 0	40	6.4038818E 43	1.0451134E 47
20	8.5582713E 25	2.7934197E 29	41	8.9994839E 46	0. E 0

I	A(I)	B(I)
42	1.8133917E 45	2.8185290E 48
43	2.4018825E 48	0. E 0
44	4.5768562E 46	6.7903902E 49
45	5.7266917E 49	0. E 0
46	1.0339997E 48	1.4673802E 51
47	1.2247012E 51	0. E 0
48	2.0990860E 49	2.8547570E 52
49	2.3579161E 52	0. E 0
50	3.8426127E 50	5.0169149E 53
51	4.1006853E 53	0. E 0
52	6.3636675E 51	7.9888502E 54
53	6.4617362E 54	0. E 0
54	9.3621262E 52	1.1559547E 56
55	9.2519264E 55	0. E 0
56	1.3072101E 54	1.5238335E 57
57	1.2067941E 57	0. E 0
58	1.6299175E 55	1.8345002E 58
59	1.4374500E 58	0. E 0
60	1.8578772E 56	2.0213703E 59
61	1.5670127E 59	0. E 0
62	1.9401105E 57	2.0427494E 60
63	1.5666186E 60	0. E 0
64	1.8597451E 58	1.8969402E 61
65	1.4390995E 61	0. E 0
66	1.6394365E 59	1.6215523E 62
67	1.2168092E 62	0. E 0
68	1.3313474E 60	1.2780935E 63
69	9.4857537E 62	0. E 0
70	9.9754578E 60	9.3028268E 63
71	6.8281505E 63	0. E 0
72	6.9065492E 61	6.2619379E 64
73	4.5450138E 64	0. E 0
74	4.4246065E 62	3.9032204E 65
75	2.8012078E 65	0. E 0
76	2.6262232E 63	2.2557875E 66
77	1.6005600E 66	0. E 0
78	1.4459486E 64	1.2101476E 67
79	8.4882386E 66	0. E 0
80	7.3930629E 64	6.0327392E 67
81	4.1826479E 67	0. E 0
82	3.5139865E 65	2.7974760E 68
83	1.9169514E 68	0. E 0
84	1.5541877E 66	1.2078259E 69
85	8.1791022E 68	0. E 0
86	6.4022087E 66	4.8597229E 69
87	3.2517434E 69	0. E 0
88	2.4583727E 67	1.8236655E 70
89	1.2055869E 70	0. E 0
90	8.8064485E 67	6.3876106E 70
91	4.1714176E 70	0. E 0
92	2.9451602E 68	2.0897832E 71
93	1.3479691E 71	0. E 0
94	9.2017323E 68	6.3903092E 71
95	4.0707431E 71	0. E 0
96	2.6875653E 69	1.8275443E 72
97	1.1495575E 72	0. E 0
98	7.3423172E 69	4.8908821E 72
99	3.0373641E 72	0. E 0
100	1.8772780E 70	1.2254871E 73
101	7.5127550E 72	0. E 0
102	4.4943151E 70	2.8763616E 73
103	1.7403907E 73	0. E 0
104	1.0079483E 71	6.3268140E 73
105	3.7777310E 73	0. E 0
106	2.1185374E 71	1.3046992E 74
107	7.6864691E 73	0. E 0
108	4.1746880E 71	2.5233669E 74
109	1.4665367E 74	0. E 0
110	7.7152997E 71	4.5786796E 74
111	2.6246551E 74	0. E 0
112	1.3376977E 72	7.7968667E 74
113	4.4075025E 74	0. E 0
114	2.1765068E 72	1.2463365E 75
115	6.9465133E 74	0. E 0
116	3.3240289E 72	1.8706259E 75
117	1.0277646E 75	0. E 0
118	4.7661068E 72	2.6367071E 75
119	1.4277669E 75	0. E 0
120	6.4170356E 72	3.4908672E 75
121	1.8626372E 75	0. E 0
122	8.1141027E 72	4.3417099E 75
123	2.2822504E 75	0. E 0
124	9.6367806E 72	5.0732986E 75
125	2.6266704E 75	0. E 0
126	1.0750917E 73	5.5699987E 75
127	2.8397818E 75	0. E 0
128	1.1266896E 73	5.7461170E 75
129	2.8841506E 75	0. E 0
130	1.1092216E 73	5.5699987E 75
131	2.7517422E 75	0. E 0
132	1.0258508E 73	5.0732986E 75
133	2.4662912E 75	0. E 0
134	8.9122109E 72	4.3417099E 75
135	2.0763688E 75	0. E 0
136	7.2726403E 72	3.4908673E 75
137	1.6419179E 75	0. E 0
138	5.5739216E 72	2.6367071E 75
139	1.2193689E 75	0. E 0
140	4.0117590E 72	1.8706259E 75
141	8.5033846E 74	0. E 0
142	2.7110875E 72	1.2463365E 75
143	5.5672915E 74	0. E 0
144	1.7198970E 72	7.7968667E 74
145	3.4213695E 74	0. E 0
146	1.0240306E 72	4.5786796E 74
147	1.9731223E 74	0. E 0
148	5.7208687E 71	2.5233669E 74
149	1.0675433E 74	0. E 0
150	2.9979303E 71	1.3046992E 74
151	5.4170021E 73	0. E 0
152	1.4731552E 71	6.3268140E 73
153	2.5770537E 73	0. E 0
154	6.7855346E 70	2.8763616E 73
155	1.1489789E 73	0. E 0

I	A(I)	B(I)	I	A(I)	B(I)
156	2.9285538E 70	1.2254871E 73	207	9.8077582E 52	0. E 0
157	4.7988966E 72	0. E 0	208	9.0960398E 49	2.8547570E 52
158	1.1837613E 70	4.8908821E 72	209	5.3573898E 51	0. E 0
159	1.8767701E 72	0. E 0	210	4.7204331E 48	1.4673802E 51
160	4.4792754E 69	1.8275443E 72	211	2.6389042E 50	0. E 0
161	6.8691495E 71	0. E 0	212	2.2052125E 47	6.7903902E 49
162	1.5858304E 69	6.3903092E 71	213	1.1680222E 49	0. E 0
163	2.3516917E 71	0. E 0	214	9.2396629E 45	2.8185289E 48
164	5.2500681E 68	2.0897832E 71	215	4.6276072E 47	0. E 0
165	7.5264015E 70	0. E 0	216	3.4580961E 44	1.0451134E 47
166	1.6243005E 68	6.3876106E 70	217	1.6341407E 46	0. E 0
167	2.2503375E 70	0. E 0	218	1.1509280E 43	3.4464487E 45
168	4.6932571E 67	1.8236655E 70	219	5.1192119E 44	0. E 0
169	6.2815060E 69	0. E 0	220	3.3894785E 41	1.0057507E 44
170	1.2655528E 67	4.8597229E 69	221	1.4158153E 43	0. E 0
171	1.6357455E 69	0. E 0	222	8.7839228E 39	2.5829479E 42
172	3.1823844E 66	1.2078258E 69	223	3.4324687E 41	0. E 0
173	3.9706352E 68	0. E 0	224	1.9907813E 38	5.8017053E 40
174	7.4565080E 65	2.7974760E 68	225	7.2560499E 39	0. E 0
175	8.9769303E 67	0. E 0	226	3.9183633E 36	1.1318174E 39
176	1.6264737E 65	6.0327392E 67	227	1.3270130E 38	0. E 0
177	1.8885307E 67	0. E 0	228	6.6449186E 34	1.9025451E 37
178	3.2997287E 64	1.2101476E 67	229	2.0818708E 36	0. E 0
179	3.6933887E 66	0. E 0	230	9.6214638E 32	2.7308223E 35
180	6.2200022E 63	2.2557875E 66	231	2.7746685E 34	0. E 0
181	6.7077510E 65	0. E 0	232	1.1771036E 31	3.3121260E 33
182	1.0882140E 63	3.9032204E 65	233	3.1063202E 32	0. E 0
183	1.1300396E 65	0. E 0	234	1.2020147E 29	3.3533136E 31
184	1.7650070E 62	6.2619379E 64	235	2.8827637E 30	0. E 0
185	1.7638205E 64	0. E 0	236	1.0098760E 27	2.7934197E 29
186	2.6506216E 61	9.3028266E 63	237	2.1830452E 28	0. E 0
187	2.5474225E 63	0. E 0	238	6.8610615E 24	1.8818911E 27
188	3.6807840E 60	1.2780935E 63	239	1.3235738E 26	0. E 0
189	3.3996553E 62	0. E 0	240	3.6909477E 22	1.0039377E 25
190	4.7195902E 59	1.6215523E 62	241	6.2761425E 23	0. E 0
191	4.1861436E 61	0. E 0	242	1.5315136E 20	4.1312895E 22
192	5.3792351E 58	1.8969402E 61	243	2.2597731E 21	0. E 0
193	4.7484212E 60	0. E 0	244	4.7399160E 17	1.2681218E 20
194	6.0706685E 57	2.0427494E 60	245	5.9453675E 18	0. E 0
195	4.9533521E 59	0. E 0	246	1.0466188E 15	2.7773698E 17
196	6.0690656E 56	2.0213703E 59	247	1.0850651E 16	0. E 0
197	4.7431509E 58	0. E 0	248	1.5502510E 12	4.0806335E 14
198	5.5642010E 55	1.8345001E 58	249	1.2753386E 13	0. E 0
199	4.1609579E 57	0. E 0	250	1.4055088E 9	3.6709350E 11
200	4.6686078E 54	1.5238335E 57	251	8.6044343E 9	0. E 0
201	3.3369629E 56	0. E 0	252	6.8008495E 5	1.7410174E 8
202	3.5769435E 53	1.1559547E 56	253	2.7203397E 6	0. E 0
203	2.4408386E 55	0. E 0	254	0. E 0	3.2639999E 4
204	2.4965158E 52	7.9888502E 54	255	2.5600000E 2	0. E 0
205	1.6243184E 54	0. E 0	256	1.0000000E 0	0. E 0
206	1.5831564E 51	5.0169148E 53			

N = 512

I	A(I)	B(I)	I	A(I)	B(I)
0	0. E 0	0. E 0	55	4.3306802E 73	0. E 0
1	0. E 0	0. E 0	56	6.9292096E 71	3.2373267E 75
2	0. E 0	1.3081599E 5	57	2.9282839E 75	0. E 0
3	1.7373998E 5	0. E 0	58	4.5039861E 73	2.0317015E 77
4	4.3434996E 4	2.8243171E 9	59	1.8286166E 77	0. E 0
5	3.3582895E 9	0. E 0	60	2.7068929E 75	1.1803496E 79
6	5.5603741E 8	2.4247608E 13	61	1.0571040E 79	0. E 0
7	2.7346614E 13	0. E 0	62	1.5076653E 77	6.3621533E 80
8	3.3831574E 12	1.1064268E 17	63	5.6697248E 80	0. E 0
9	1.2082597E 17	0. E 0	64	7.7988080E 78	3.1881771E 82
10	1.1912086E 16	3.1165830E 20	65	2.8271974E 82	0. E 0
11	3.3280680E 20	0. E 0	66	3.7542548E 80	1.4082321E 84
12	2.7235578E 19	5.9380820E 23	67	1.3134896E 84	0. E 0
13	6.2345026E 23	0. E 0	68	1.0849998E 82	6.4830863E 85
14	4.3559463E 22	8.1403924E 26	69	5.6926876E 85	0. E 0
15	8.4311951E 26	0. E 0	70	7.0636396E 83	2.6401058E 87
16	5.1339182E 25	8.3949828E 29	71	2.3068573E 87	0. E 0
17	8.5960509E 29	0. E 0	72	2.7703395E 85	1.0060796E 89
18	4.6341159E 28	6.7357405E 32	73	8.7529652E 88	0. E 0
19	6.8289376E 32	0. E 0	74	1.0181099E 87	3.5995983E 90
20	3.3000060E 31	4.3169358E 35	75	3.1144560E 90	0. E 0
21	4.3381123E 35	0. E 0	76	3.5112146E 88	1.2067451E 92
22	1.8980691E 34	2.2572528E 38	77	1.0407016E 92	0. E 0
23	2.2501420E 38	0. E 0	78	1.1379649E 90	3.8170262E 93
24	8.9880698E 36	9.7981942E 40	79	3.2702504E 93	0. E 0
25	9.6949151E 40	0. E 0	80	3.4704420E 91	1.1349733E 95
26	3.5601143E 39	3.5824608E 43	81	9.6760552E 94	0. E 0
27	3.5200765E 43	0. E 0	82	9.9717369E 92	3.1816163E 96
28	1.1953767E 42	1.1169599E 46	83	2.6990879E 96	0. E 0
29	1.0902907E 46	0. E 0	84	2.7027420E 94	8.4181403E 97
30	3.4414484E 44	3.0013101E 48	85	7.1062186E 97	0. E 0
31	2.9112446E 48	0. E 0	86	6.9179596E 95	2.1046040E 99
32	8.5792714E 46	7.0144122E 50	87	1.7678365E 99	0. E 0
33	6.7628054E 50	0. E 0	88	1.6740130E 97	4.9769927E100
34	1.8679412E 49	1.4373917E 53	89	4.1599067E100	0. E 0
35	1.3777299E 53	0. E 0	90	3.8334725E 98	1.1143990E102
36	3.5790068E 51	2.6010630E 55	91	9.2682427E101	0. E 0
37	2.4789275E 55	0. E 0	92	8.3157798E 99	2.3648631E103
38	6.0751756E 53	4.1827903E 57	93	1.9570307E103	0. E 0
39	3.9642562E 57	0. E 0	94	1.7103914E101	4.7605651E104
40	9.1907324E 55	6.0114741E 59	95	3.9199403E104	0. E 0
41	5.6664133E 59	0. E 0	96	3.3385383E102	9.0986297E105
42	1.2458597E 58	7.7608773E 61	97	7.4545314E105	0. E 0
43	7.2763053E 61	0. E 0	98	6.1894893E103	1.6524168E107
44	1.5206252E 60	9.0419139E 63	99	1.3470417E107	0. E 0
45	8.4327238E 63	0. E 0	100	1.0907969E105	2.8538739E108
46	1.3784938E 62	9.5466884E 65	101	2.3147644E108	0. E 0
47	8.8572075E 65	0. E 0	102	1.8287804E106	4.6906576E109
48	1.6822978E 64	9.1696444E 67	103	3.7855405E109	0. E 0
49	8.4636513E 67	0. E 0	104	2.9189814E107	7.3432590E110
50	1.5366130E 66	8.0405427E 69	105	5.8960471E110	0. E 0
51	7.3836749E 69	0. E 0	106	4.4387761E108	1.0955904E112
52	1.2834153E 68	6.4573559E 71	107	8.7520406E111	0. E 0
53	5.8998541E 71	0. E 0	108	6.4350903E109	1.5589125E113
54	9.8323764E 69	4.7638227E 73	109	1.2389794E113	0. E 0

I	A(I)	B(I)	I	A(I)	B(I)
110	8.9000162E110	2.1168444E114	167	3.1214521E138	0. E 0
111	1.6738938E114	0. E 0	168	1.2566578E138	1.9573466E139
112	1.1750207E112	2.7448485E115	169	1.3203155E139	0. E 0
113	2.1592304E115	0. E 0	170	5.2232732E136	8.0366786E139
114	1.4817739E113	3.4006971E116	171	5.3905404E139	0. E 0
115	2.6613693E116	0. E 0	172	2.0954542E137	3.1874295E140
116	1.7858822E114	4.0279649E117	173	2.1247691E140	0. E 0
117	3.1359616E117	0. E 0	174	8.1167513E137	1.2204608E141
118	2.0582647E115	4.5636263E118	175	8.0873110E140	0. E 0
119	3.5345496E118	0. E 0	176	3.0362778E138	4.5135652E141
120	2.2696612E116	4.9484665E119	177	2.9729978E141	0. E 0
121	3.8126280E119	0. E 0	178	1.0970834E139	1.6125400E142
122	2.3958361E117	5.1379316E120	179	1.0557600E142	0. E 0
123	3.9378807E120	0. E 0	180	3.8296345E139	5.5664162E142
124	2.4221803E118	5.1106454E121	181	3.6229903E142	0. E 0
125	3.8963756E121	0. E 0	182	1.2917317E140	1.8569138E143
126	2.3464831E119	4.8723431E122	183	1.2010498E143	0. E 0
127	3.6950815E122	0. E 0	184	4.2107521E140	5.9873230E143
128	2.1791710E120	4.4542254E123	185	3.8468962E143	0. E 0
129	3.3600788E123	0. E 0	186	1.3267579E141	1.8662490E144
130	1.9409825E121	3.9063318E124	187	1.1923172E144	0. E 0
131	2.9310777E124	0. E 0	188	4.0414582E141	5.6243339E144
132	1.6588049E122	3.2878518E125	189	3.5710520E144	0. E 0
133	2.4538103E125	0. E 0	190	1.1903261E142	1.6390916E145
134	1.3607945E123	2.6569206E126	191	1.0342252E145	0. E 0
135	1.9722744E126	0. E 0	192	3.3903138E142	4.6198674E145
136	1.0719810E124	2.0622392E127	193	2.8967565E145	0. E 0
137	1.5225692E127	0. E 0	194	9.3394650E142	1.2595375E146
138	8.1123506E124	1.5380076E128	195	7.8477958E145	0. E 0
139	1.1293661E128	0. E 0	196	2.4886970E143	3.3220550E146
140	5.8997452E125	1.1025443E129	197	2.0567465E146	0. E 0
141	8.0519216E128	0. E 0	198	6.4157487E143	8.4776020E146
142	4.1248004E126	7.5998576E129	199	5.2151573E146	0. E 0
143	5.5198179E129	0. E 0	200	1.6003061E144	2.0934563E147
144	2.7733624E127	5.0388915E130	201	1.2795626E147	0. E 0
145	3.6396417E130	0. E 0	202	3.8627089E144	5.0030110E147
146	1.7938661E128	3.2146079E131	203	3.0381901E147	0. E 0
147	2.3091082E131	0. E 0	204	9.0232862E144	1.1572453E148
148	1.1165900E129	1.9738897E132	205	6.9819553E147	0. E 0
149	1.4100026E132	0. E 0	206	2.0401860E145	2.5911552E148
150	6.6904260E129	1.1669530E133	207	1.5530853E148	0. E 0
151	8.2893234E132	0. E 0	208	4.4653292E145	5.6166971E148
152	3.8601246E130	6.6442903E133	209	3.3443753E148	0. E 0
153	4.6932182E133	0. E 0	210	9.4615017E145	1.1787769E149
154	2.1451775E131	3.6444617E134	211	6.9723522E148	0. E 0
155	2.5597616E134	0. E 0	212	1.9410278E146	2.3954479E149
156	1.1485821E132	1.9263194E135	213	1.4074385E149	0. E 0
157	1.3453207E135	0. E 0	214	3.8557486E146	4.7139590E149
158	5.9267437E132	9.8140870E135	215	2.7510884E149	0. E 0
159	6.8150017E135	0. E 0	216	7.4170155E146	8.9839287E149
160	2.9480842E133	4.8207072E136	217	5.2076510E149	0. E 0
161	3.3283760E136	0. E 0	218	1.3817497E147	1.6583029E150
162	1.4139816E134	2.2835979E137	219	9.5472159E149	0. E 0
163	1.5675941E137	0. E 0	220	2.4931234E147	2.9649137E150
164	6.5408565E134	1.0434740E138	221	1.6952842E150	0. E 0
165	7.1215509E137	0. E 0	222	4.3571709E147	5.1350253E150
166	2.9188753E135	4.6004288E138	223	2.9158867E150	0. E 0

I	A(I)	B(I)	I	A(I)	B(I)
224	7.5763083E147	8.6156005E150	281	2.2585627E151	0. E 0
225	4.6583724E150	0. E 0	282	3.6275946E148	3.3655852E151
226	1.2097244E148	1.4004543E151	283	1.5142781E151	0. E 0
227	7.8420711E150	0. E 0	284	2.3941233E148	2.2055614E151
228	1.9220426E148	2.2055614E151	285	9.8569493E150	0. E 0
229	1.2263560E151	0. E 0	286	1.5308901E148	1.4004543E151
230	2.9586763E148	3.3655852E151	287	6.1911771E150	0. E 0
231	1.8581140E151	0. E 0	288	9.4839021E147	8.6156005E150
232	4.4127879E148	4.9764087E151	289	3.7750170E150	0. E 0
233	2.7278507E151	0. E 0	290	5.6917999E147	5.1350254E150
234	6.3772195E148	7.1302784E151	291	2.2298238E150	0. E 0
235	3.8804486E151	0. E 0	292	3.3090547E147	2.9649137E150
236	8.9304268E148	9.9003644E151	293	1.2758497E150	0. E 0
237	5.3490337E151	0. E 0	294	1.8634507E147	1.6583029E150
238	1.2118632E149	1.3321945E152	295	7.0709055E149	0. E 0
239	7.1452584E151	0. E 0	296	1.0164058E147	8.9839287E149
240	1.5936464E149	1.7372875E152	297	3.7954640E149	0. E 0
241	9.2496493E151	0. E 0	298	5.3692201E146	4.7139590E149
242	2.0309623E149	2.1957226E152	299	1.9730342E149	0. E 0
243	1.1604089E152	0. E 0	300	2.7467374E146	2.3954479E149
244	2.5083968E149	2.6896600E152	301	9.9322580E148	0. E 0
245	1.4108714E152	0. E 0	302	1.3606540E146	1.1787769E149
246	3.0025234E149	3.1933185E152	303	4.8413335E148	0. E 0
247	1.6625120E152	0. E 0	304	6.5262504E145	5.6166971E148
248	3.4832274E149	3.6746934E152	305	2.2646156E148	0. E 0
249	1.6986603E152	0. E 0	306	3.0305678E145	2.5911552E148
250	3.9164310E149	4.0986556E152	307	1.0438974E148	0. E 0
251	2.1016266E152	0. E 0	308	1.3623393E145	1.1572453E148
252	4.2679421E149	4.4310734E152	309	4.6168271E147	0. E 0
253	2.2546622E152	0. E 0	310	5.9279196E144	5.0030110E147
254	4.5078616E149	4.6433115E152	311	1.9763406E147	0. E 0
255	2.3444084E152	0. E 0	312	2.4964775E144	2.0934563E147
256	4.6147639E149	4.7162899E152	313	6.1677387E146	0. E 0
257	2.3627239E152	0. E 0	314	1.0174470E144	8.4776020E146
258	4.5788516E149	4.6433115E152	315	3.2824600E146	0. E 0
259	2.3079213E152	0. E 0	316	4.0123891E143	3.3220549E146
260	4.4034323E149	4.4310734E152	317	1.2732550E146	0. E 0
261	2.1850233E152	0. E 0	318	1.5309019E143	1.2595375E146
262	4.1044197E149	4.0986556E152	319	4.7781171E145	0. E 0
263	2.0050046E152	0. E 0	320	5.6505229E142	4.6198674E145
264	3.7079517E149	3.6746935E152	321	1.7344676E145	0. E 0
265	1.7831763E152	0. E 0	322	2.0172894E142	1.6390916E145
266	3.2466310E149	3.1933185E152	323	6.0895313E144	0. E 0
267	1.5370449E152	0. E 0	324	6.9650664E141	5.6243339E144
268	2.7551244E149	2.6896600E152	325	2.0675105E144	0. E 0
269	1.2840571E152	0. E 0	326	2.3253929E141	1.8662490E144
270	2.2659497E149	2.1957226E152	327	6.7872390E143	0. E 0
271	1.0396281E152	0. E 0	328	7.5061232E140	5.9873230E143
272	1.8061326E149	1.7372875E152	329	2.1540364E143	0. E 0
273	8.1574866E151	0. E 0	330	2.3421509E140	1.8569138E143
274	1.3951703E149	1.3321945E152	331	6.6078123E142	0. E 0
275	6.2030669E151	0. E 0	332	7.0035481E139	5.5664162E142
276	1.0444058E149	9.9003644E151	333	1.9590017E142	0. E 0
277	4.5710254E151	0. E 0	334	2.0585723E139	1.6125400E142
278	7.5763550E148	7.1302784E151	335	5.6118928E141	0. E 0
279	3.2640857E151	0. E 0	336	5.7965305E138	4.5135651E141
280	5.3257785E148	4.9764087E151	337	1.5531147E141	0. E 0

I	A(I)	B(I)	I	A(I)	B(I)
338	1.5767022E138	1.2204606E141	395	1.0523828E118	0. E 0
339	4.1518620E140	0. E 0	396	6.0966325E114	4.0279649E117
340	4.1421769E137	3.1674295E140	397	9.1310314E116	0. E 0
341	1.0716278E140	0. E 0	398	5.1732107E113	3.4006971E116
342	1.0507997E137	8.0386786E139	399	7.5760613E115	0. E 0
343	2.6716660E139	0. E 0	400	4.1965027E112	2.7448485E115
344	2.5735660E136	1.9573466E139	401	6.0076085E114	0. E 0
345	6.4286276E138	0. E 0	402	3.2525513E111	2.1168444E114
346	6.0839208E135	4.6004288E138	403	4.5503153E113	0. E 0
347	1.4929332E138	0. E 0	404	2.4072005E110	1.5589125E113
348	1.3879378E135	1.0434740E138	405	3.2900312E112	0. E 0
349	3.3454325E137	0. E 0	406	1.7001349E109	1.0955904E112
350	3.0548987E134	2.2835979E137	407	2.2693597E111	0. E 0
351	7.2319323E136	0. E 0	408	1.1451388E108	7.3432590E110
352	6.4857852E133	4.8207072E136	409	1.4923345E110	0. E 0
353	1.5077988E136	0. E 0	410	7.3509602E106	4.6908576E109
354	1.3278906E133	9.8140870E135	411	9.3495026E108	0. E 0
355	3.0311870E135	0. E 0	412	4.4940635E105	2.8538759E108
356	2.6211233E132	1.9263194E135	413	5.5765873E107	0. E 0
357	5.8742414E134	0. E 0	414	2.6147434E104	1.6524168E107
358	4.9868413E131	3.6444616E134	415	3.1642756E106	0. E 0
359	1.0970999E134	0. E 0	416	1.4466999E103	9.0986296E105
360	9.1424003E130	6.6442903E133	417	1.7067536E105	0. E 0
361	1.9741383E133	0. E 0	418	7.6057631E101	4.7605651E104
362	1.6146228E130	1.1669530E133	419	8.7438965E103	0. E 0
363	3.4215537E132	0. E 0	420	3.7963343E100	2.3648631E103
364	2.7462080E129	1.9738897E132	421	4.2511609E102	0. E 0
365	5.7102721E131	0. E 0	422	1.7974727E 99	1.1143990E102
366	4.4969518E128	3.2146078E131	423	1.9597109E101	0. E 0
367	9.1737429E130	0. E 0	424	8.0656990E 97	4.9769927E100
368	7.0874819E127	5.0388913E130	425	8.5576332E 99	0. E 0
369	1.4182629E130	0. E 0	426	3.4268032E 96	2.1046040E 99
370	1.0747719E127	7.5998574E129	427	3.5364544E 98	0. E 0
371	2.1093405E129	0. E 0	428	1.3771113E 95	8.4181403E 97
372	1.5676465E126	1.1025443E129	429	1.3816240E 97	0. E 0
373	3.0169682E128	0. E 0	430	5.2290814E 93	3.1816162E 96
374	2.1985645E125	1.5380075E128	431	5.0974315E 95	0. E 0
375	4.1483747E127	0. E 0	432	1.8740388E 92	1.1349733E 95
376	2.9637122E124	2.0622392E127	433	1.7740290E 94	0. E 0
377	5.4816615E126	0. E 0	434	6.3317537E 90	3.8170262E 93
378	3.8386589E123	2.6569206E126	435	5.8170153E 92	0. E 0
379	6.9584347E125	0. E 0	436	2.0143283E 89	1.2087451E 92
380	4.7753473E122	3.2878517E125	437	1.7948336E 91	0. E 0
381	8.4821961E124	0. E 0	438	6.0261105E 87	3.5995983E 90
382	5.7035023E121	3.9063317E124	439	5.2042364E 89	0. E 0
383	9.9249578E123	0. E 0	440	1.6929852E 86	1.0066798E 89
384	6.5375131E120	4.4542253E123	441	1.4160895E 88	0. E 0
385	1.1142752E123	0. E 0	442	4.4601838E 84	2.6401058E 87
386	7.1884325E119	4.8723431E122	443	3.6106200E 86	0. E 0
387	1.1998111E122	0. E 0	444	1.1002057E 83	6.4830863E 85
388	7.5790806E118	5.1106454E121	445	8.6128860E 84	0. E 0
389	1.2385003E121	0. E 0	446	2.5369661E 81	1.4882321E 84
390	7.6588206E117	5.1379316E120	447	1.9189700E 83	0. E 0
391	1.2250143E120	0. E 0	448	5.4592074E 79	3.1881770E 82
392	7.4142263E116	4.9484663E119	449	3.9863156E 81	0. E 0
393	1.1604840E119	0. E 0	450	1.0942731E 78	6.3621531E 80
394	6.6725111E115	4.5636263E118	451	7.7062088E 79	0. E 0

I	A(I)	B(I)	I	A(I)	B(I)
452	2.0391926E 76	1.1803496E 79	483	1.7587868E 47	0. E 0
453	1.3835748E 78	0. E 0	484	2.0662941E 43	1.1169599E 46
454	3.5255339E 74	2.0317014E 77	485	6.1090406E 44	0. E 0
455	2.3021011E 76	0. E 0	486	6.6546752E 40	3.5824607E 43
456	5.6423564E 72	3.2373267E 75	487	1.6194010E 42	0. E 0
457	3.5416600E 74	0. E 0	488	1.8275742E 38	9.7981942E 40
458	8.3393117E 70	4.7638227E 73	489	4.5933263E 39	0. E 0
459	5.0254795E 72	0. E 0	490	4.2275176E 35	2.2572528E 38
460	1.1353289E 69	6.4573559E 71	491	9.6999449E 36	0. E 0
461	6.5596719E 70	0. E 0	492	8.1180149E 32	4.3169356E 35
462	1.4198304E 67	8.0405427E 69	493	1.6864301E 34	0. E 0
463	7.8537187E 68	0. E 0	494	1.2718076E 30	6.7357403E 32
464	1.6262211E 65	9.1696444E 67	495	2.3681929E 31	0. E 0
465	8.5982420E 66	0. E 0	496	1.5915145E 27	8.3949828E 29
466	1.7003871E 63	9.5466883E 65	497	2.6235870E 28	0. E 0
467	8.5787199E 64	0. E 0	498	1.5494723E 24	8.1403923E 26
468	1.6173922E 61	9.0419139E 63	499	2.2260019E 25	0. E 0
469	7.7717890E 62	0. E 0	500	1.1348157E 21	5.9380820E 23
470	1.3941763E 59	7.7608772E 61	501	1.3917978E 22	0. E 0
471	6.3674290E 60	0. E 0	502	5.9798674E 17	3.1165830E 20
472	1.0845063E 57	6.0114741E 59	503	6.0872894E 18	0. E 0
473	4.6972218E 58	0. E 0	504	2.1313891E 14	1.1064267E 17
474	7.5779822E 54	4.1827903E 57	505	1.7288389E 15	0. E 0
475	3.1048879E 56	0. E 0	506	4.6892488E 10	2.4247608E 13
476	4.7322423E 52	2.6010630E 55	507	2.6415717E 11	0. E 0
477	1.8291349E 54	0. E 0	508	5.5162444E 6	2.8243171E 9
478	2.3261057E 50	1.4373917E 53	509	2.2064978E 7	0. E 0
479	9.3464666E 51	0. E 0	510	0. E 0	1.3081599E 5
480	1.2868906E 48	7.0144122E 50	511	5.1199998E 2	0. E 0
481	4.3845603E 49	0. E 0	512	1.0000000E 0	0. E 0
482	5.5292603E 45	3.0013101E 48			

ERROR RATE EQUATION COEFFICIENTS

HAMMING SINGLE ERROR CORRECTING /DOUBLE ERROR DETECTING CODES

DETECTION WITHOUT CORRECTION OCCURS IF CHECK WORD IS NON-ZERO AND OVERALL PARITY CHECK IS SATISFIED - OR - IF CHECK WORD IS ZERO AND OVERALL PARITY CHECK IS NOT SATISFIED.

I	A(I)	B(I)	I	A(I)	B(I)
N = 8					
0	0. E 0	0. E 0	5	2.3999999E 1	7.0000000E 0
1	0. E 0	1.0000000E 0	6	0. E 0	2.7999999E 1
2	0. E 0	2.7999999E 1	7	6.9999999E 0	1.0000000E 0
3	2.4999999E 1	7.0000000E 0	8	1.0000000E 0	0. E 0
4	6.9999999E 0	5.5999998E 1			
N = 16					
0	0. E 0	0. E 0	9	5.8799994E 3	7.1499996E 2
1	0. E 0	1.0000000E 0	10	2.7999997E 2	7.5599993E 3
2	0. E 0	1.2000000E 2	11	2.7509998E 3	2.7299998E 2
3	1.3299999E 2	3.4999998E 1	12	1.0500000E 2	1.6799998E 3
4	3.4999998E 1	1.6799998E 3	13	3.9199998E 2	3.4999998E 1
5	1.3439998E 3	2.7299998E 2	14	0. E 0	1.2000000E 2
6	1.6799999E 2	7.5599995E 3	15	1.5000000E 1	1.0000000E 0
7	4.8449994E 3	7.1499996E 2	16	1.0000000E 0	0. E 0
8	4.3499996E 2	1.1999999E 4			
N = 32					
0	0. E 0	0. E 0	17	2.9003106E 8	1.7678832E 7
1	0. E 0	1.0000000E 0	18	8.2807187E 6	4.5671424E 8
2	0. E 0	4.9599999E 2	19	1.9778178E 8	1.0855423E 7
3	6.0499997E 2	1.5499999E 2	20	4.4148644E 6	2.1872902E 8
4	1.5499999E 2	3.4719996E 4	21	8.0760564E 7	4.0320145E 6
5	3.4607997E 4	6.2929996E 3	22	1.3830958E 6	6.2500456E 7
6	5.2079994E 3	8.7841592E 5	23	1.9149192E 7	8.7652490E 5
7	7.7330492E 5	1.0518298E 5	24	2.4784497E 5	1.0187838E 7
8	8.2614992E 4	1.0187838E 7	25	2.4873677E 6	1.0518298E 5
9	8.0230790E 6	8.7652490E 5	26	2.2567998E 4	8.7841591E 5
10	6.2867992E 5	6.2500456E 7	27	1.6047498E 5	6.2929996E 3
11	4.4231882E 7	4.0320145E 6	28	1.0850000E 3	3.4719996E 4
12	2.6489185E 6	2.1872902E 8	29	4.1999997E 3	1.5499999E 2
13	1.3873633E 8	1.0855423E 7	30	0. E 0	4.9599999E 2
14	6.4405589E 6	4.5671424E 8	31	3.0999998E 1	1.0000000E 0
15	2.5801271E 8	1.7678832E 7	32	1.0000000E 0	0. E 0
16	9.3981135E 6	5.8228405E 8			
N = 64					
0	0. E 0	0. E 0	8	8.6492779E 6	4.3569705E 9
1	0. E 0	1.0000000E 0	9	4.1216123E 9	4.3032157E 8
2	0. E 0	2.0159999E 3	10	3.6977662E 8	1.4910661E 11
3	2.5729998E 3	6.5099999E 2	11	1.3343363E 11	1.1618682E 10
4	6.5099998E 2	6.2495995E 5	12	9.6218882E 9	3.2328973E 12
5	6.8596794E 5	1.1913300E 5	13	2.7486034E 12	2.0526338E 11
6	1.0936798E 5	7.3807768E 7	14	1.6356852E 11	4.7107946E 13
7	7.4475848E 7	9.7065020E 6	15	3.8127218E 13	2.4924837E 12

I	A(I)	B(I)	I	A(I)	B(I)
16	1.9083105E 12	4.8089357E 14	41	9.1879964E 16	2.2925216E 15
17	3.7077297E 14	2.1552653E 13	42	8.2387484E 14	7.9091995E 16
18	1.5827722E 13	3.5454114E 15	43	2.6967078E 16	6.4231226E 14
19	2.6036151E 15	1.3624805E 14	44	2.1075875E 14	1.9313161E 16
20	9.5799435E 13	1.9313161E 16	45	5.9800123E 15	1.3624805E 14
21	1.3498593E 16	6.4231226E 14	46	4.0448622E 13	3.5454114E 15
22	4.3155349E 14	7.9091995E 16	47	9.8704418E 14	2.1552653E 13
23	5.2548893E 16	2.2925216E 15	48	5.7249314E 12	4.8089357E 14
24	1.4686466E 15	2.4673263E 17	49	1.1889926E 14	2.4924837E 12
25	1.5557862E 17	6.2662255E 15	50	5.8417331E 11	4.7107946E 13
26	3.8184809E 15	5.9215830E 17	51	1.0182989E 13	2.0526338E 11
27	3.5366096E 17	1.3228698E 16	52	4.1694847E 10	3.2328973E 12
28	7.6478410E 15	1.1012890E 18	53	5.9854331E 11	1.1618682E 10
29	6.2150948E 17	2.1700277E 16	54	1.9967937E 9	1.4910661E 11
30	1.1867338E 16	1.5949704E 18	55	2.2988645E 10	4.3032157E 8
31	8.4819571E 17	2.7767021E 16	56	6.0544945E 7	4.3569705E 9
32	1.4317370E 16	1.8039887E 18	57	5.3703375E 8	9.7065020E 6
33	9.0112658E 17	2.7767021E 16	58	1.0572238E 6	7.3807767E 7
34	1.3449650E 16	1.5949704E 18	59	6.8194103E 6	1.1913300E 5
35	7.4560795E 17	2.1700277E 16	60	9.7649993E 3	6.2495995E 5
36	9.8329383E 15	1.1012890E 18	61	3.8439997E 4	6.5099999E 2
37	4.7974699E 17	1.3228698E 16	62	0. E 0	2.0159999E 3
38	5.5808567E 15	5.9215830E 17	63	6.3000000E 1	1.0000000E 0
39	2.3919356E 17	6.2662255E 15	64	1.0000000E 0	0. E 0
40	2.4477445E 15	2.4673263E 17			

N = 128

0	0. E 0	0. E 0	30	2.8237916E 26	1.5301185E 29
1	0. E 0	1.0000000E 0	31	1.1911359E 29	3.8087787E 27
2	0. E 0	8.1279998E 3	32	2.8863401E 27	1.4662611E 30
3	1.0604999E 4	2.6669999E 3	33	1.1159642E 30	3.3586505E 28
4	2.6669998E 3	1.0582655E 7	34	2.4927482E 28	1.1918272E 31
5	1.2155470E 7	2.0669248E 6	35	8.8666862E 30	2.5203990E 29
6	1.9842478E 6	5.3812798E 9	36	1.8312272E 29	8.2690103E 31
7	5.7867826E 9	7.3848267E 8	37	6.0116510E 31	1.6189592E 30
8	6.9813629E 8	1.4185322E 12	38	1.1509788E 30	4.9237662E 32
9	1.4578577E 12	1.4892733E 11	39	3.4969718E 32	8.9468800E 30
10	1.3845528E 11	2.2507387E 14	40	6.2208773E 30	2.5281645E 33
11	2.2323367E 14	1.9011250E 13	41	1.7534680E 33	4.2726806E 31
12	1.7377477E 13	2.3540680E 16	42	2.9040875E 31	1.1240201E 34
13	2.2651754E 16	1.6539787E 15	43	7.6100919E 33	1.7701105E 32
14	1.4859963E 15	1.7254542E 18	44	1.1754639E 32	4.3428050E 34
15	1.6157876E 18	1.0325552E 17	45	2.8689156E 34	6.3831256E 32
16	9.1155268E 16	9.2613750E 19	46	4.1390579E 32	1.4627070E 35
17	8.4565274E 19	4.8044187E 18	47	9.4238421E 34	2.0094149E 33
18	4.1663318E 18	3.7626604E 21	48	1.2715828E 33	4.3064273E 35
19	3.3541406E 21	1.7152617E 20	49	2.7044923E 35	5.5361431E 33
20	1.4606526E 20	1.1872184E 23	50	3.4168382E 33	1.1108824E 36
21	1.0340167E 23	4.8076334E 21	51	6.7966136E 35	1.3377926E 34
22	4.0188810E 21	2.9695877E 24	52	8.0476589E 33	2.5158220E 36
23	2.5282255E 24	1.0776320E 23	53	1.4986516E 36	2.8406250E 34
24	8.8399496E 22	5.9875923E 25	54	1.6644285E 34	5.0105467E 36
25	4.9844347E 25	1.9612901E 24	55	2.9041869E 36	5.3082384E 34
26	1.5782256E 24	9.8675520E 26	56	3.0273547E 34	8.7879781E 36
27	8.0328982E 26	2.9352298E 25	57	4.9527977E 36	8.7406331E 34
28	2.3160797E 25	1.3446496E 28	58	4.8483197E 34	1.3588670E 37
29	1.0704736E 28	3.6509630E 26	59	7.4412214E 36	1.2694607E 35

I	A(I)	B(I)	I	A(I)	B(I)
60	6.8431865E 34	1.8540473E 37	95	3.1495223E 30	3.3586505E 28
61	9.8572545E 36	1.6274069E 35	96	8.6590204E 27	1.4662611E 30
62	8.5184579E 34	2.2334847E 37	97	3.6460128E 29	3.8087787E 27
63	1.1519284E 37	1.8423945E 35	98	9.2243860E 26	1.5301185E 29
64	9.3559102E 34	2.3764012E 37	99	3.5662492E 28	3.6509630E 26
65	1.1879126E 37	1.8423945E 35	100	8.2717134E 25	1.3446496E 28
66	9.0680360E 34	2.2334847E 37	101	2.9244518E 27	2.9352298E 25
67	1.0810812E 37	1.6274069E 35	102	6.1915003E 24	9.8675520E 26
68	7.7556114E 34	1.8540473E 37	103	1.9923948E 26	1.9612901E 24
69	8.6809291E 36	1.2694607E 35	104	3.8306447E 23	5.9875923E 25
70	5.8514204E 34	1.3588670E 37	105	1.1157700E 25	1.0776320E 23
71	6.1478059E 36	8.7406331E 34	106	1.9363699E 22	2.9695876E 24
72	3.8923132E 34	8.7879781E 36	107	5.0716775E 23	4.8076334E 21
73	3.8372757E 36	5.3082384E 34	108	7.8875237E 20	1.1872184E 23
74	2.2808837E 34	5.0105467E 36	109	1.8429682E 22	1.7152617E 20
75	2.1089420E 36	2.8406250E 34	110	2.5460916E 19	3.7626604E 21
76	1.1761962E 34	2.5158220E 36	111	5.2559589E 20	4.8044187E 18
77	1.0193352E 36	1.3377926E 34	112	6.3808686E 17	9.2613750E 19
78	5.3302675E 33	1.1108824E 36	113	1.1497663E 19	1.0325552E 17
79	4.3264091E 35	5.5361431E 33	114	1.2100256E 16	1.7254542E 18
80	2.1193047E 33	4.3064273E 35	115	1.8740353E 17	1.6539787E 15
81	1.6095727E 35	2.0094149E 33	116	1.6798228E 14	2.3540680E 16
82	7.3783206E 32	1.4627070E 35	117	2.1911950E 15	1.9011250E 13
83	5.2376536E 34	6.3831256E 32	118	1.6337723E 12	2.2507387E 14
84	2.2440675E 32	4.3428050E 34	119	1.7455913E 13	1.4892733E 11
85	1.4870310E 34	1.7701105E 32	120	1.0472044E 10	1.4165322E 12
86	5.9464650E 31	1.1240201E 34	121	8.8000515E 10	7.3648267E 8
87	3.6728362E 33	4.2726806E 31	122	4.0346372E 7	5.3812796E 9
88	1.3685930E 31	2.5281645E 33	123	2.5034397E 8	2.0669248E 6
89	7.8655654E 32	8.9468800E 30	124	8.2676995E 4	1.0582655E 7
90	2.7260025E 30	4.9237662E 32	125	3.2810397E 5	2.6669999E 3
91	1.4549131E 32	1.6189592E 30	126	0. E 0	8.1279998E 3
92	4.6798030E 29	8.2690103E 31	127	1.2700000E 2	1.0000000E 0
93	2.3142385E 31	2.5203990E 29	128	1.0000000E 0	0. E 0
94	6.8917156E 28	1.1918272E 31			

N = 256

0	0. E 0	0. E 0	20	8.5582713E 25	2.7934197E 29
1	0. E 0	1.0000000E 0	21	2.6780949E 29	1.2310669E 28
2	0. E 0	3.2639999E 4	22	1.1300992E 28	3.3533137E 31
3	4.3052996E 4	1.0794999E 4	23	3.1748874E 31	1.3378949E 30
4	1.0794999E 4	1.7410174E 8	24	1.2176934E 30	3.3121260E 33
5	2.0444153E 8	3.4412297E 7	25	3.0986075E 33	1.2053541E 32
6	3.3732213E 7	3.6709350E 11	26	1.0876437E 32	2.7308223E 35
7	4.0709200E 11	5.1413609E 10	27	2.5254559E 35	9.1225727E 33
8	5.0008099E 10	4.0806335E 14	28	8.1604264E 33	1.9025451E 37
9	4.3630660E 14	4.4095737E 13	29	1.7398229E 37	5.8658590E 35
10	4.2545484E 13	2.7773698E 17	30	5.2013671E 35	1.1318174E 39
11	2.8915232E 17	2.4357682E 16	31	1.0237039E 39	3.2358096E 37
12	2.3311063E 16	1.2681218E 20	32	2.8439732E 37	5.8017053E 40
13	1.2925386E 20	9.3339884E 18	33	5.1911129E 40	1.5443636E 39
14	8.8599968E 18	4.1312897E 22	34	1.3452854E 39	2.5829480E 42
15	4.1361077E 22	2.6137832E 21	35	2.2865804E 42	6.4248121E 40
16	2.4606318E 21	1.0039377E 25	36	5.5464196E 40	1.0057507E 44
17	9.8939110E 24	5.5581331E 23	37	8.8098960E 43	2.3451527E 42
18	5.1890383E 23	1.8818911E 27	38	2.0062049E 42	3.4464487E 45
19	1.8282923E 27	9.2443776E 25	39	2.9873959E 45	7.5548100E 43

I	A(I)	B(I)	I	A(I)	B(I)
40	6.4038818E 43	1.0451134E 47	97	1.1450782E 72	1.1821592E 70
41	8.9649030E 46	2.1592014E 45	98	7.3423172E 69	4.8908821E 72
42	1.8133917E 45	2.8185290E 48	99	3.0255265E 72	3.0610394E 70
43	2.3926429E 48	5.5008227E 46	100	1.8772780E 70	1.2254871E 73
44	4.5768562E 46	6.7903902E 49	101	7.4834695E 72	7.4228688E 70
45	5.7046396E 49	1.2545209E 48	102	4.4943151E 70	2.8763616E 73
46	1.0339997E 48	1.4673802E 51	103	1.7336052E 73	1.6865018E 71
47	1.2199808E 51	2.5711295E 49	104	1.0079483E 71	6.3268140E 73
48	2.0990860E 49	2.8547570E 52	105	3.7629995E 73	3.5916927E 71
49	2.3488202E 52	4.7522167E 50	106	2.1185374E 71	1.3046992E 74
50	3.8426127E 50	5.0169149E 53	107	7.6564898E 73	7.1726186E 71
51	4.0848537E 53	7.9468241E 51	108	4.1746880E 71	2.5233669E 74
52	6.3636675E 51	7.9888502E 54	109	1.4608158E 74	1.3436166E 72
53	6.4367710E 54	1.2058642E 53	110	7.7152997E 71	4.5786796E 74
54	9.5621262E 52	1.1559547E 56	111	2.6144146E 74	2.3617285E 72
55	9.2161571E 55	1.6649046E 54	112	1.3376977E 72	7.7968667E 74
56	1.3072101E 54	1.5238335E 57	113	4.3903036E 74	3.8964040E 72
57	1.2021255E 57	2.0967762E 55	114	2.1765068E 72	1.2463365E 75
58	1.6299175E 55	1.8345002E 58	115	6.9194023E 74	6.0351167E 72
59	1.4318857E 58	2.4142973E 56	116	3.3240289E 72	1.8706259E 75
60	1.8578772E 56	2.0213703E 59	117	1.0237529E 75	8.7778660E 72
61	1.5609437E 59	2.5470176E 57	118	4.7661068E 72	2.6367071E 75
62	1.9401105E 57	2.0427494E 60	119	1.4221930E 75	1.1990957E 73
63	1.5605479E 60	2.4668118E 58	120	6.4170356E 72	3.4908672E 75
64	1.8597451E 58	1.8969402E 61	121	1.8553646E 75	1.5386743E 73
65	1.4335203E 61	2.1973601E 59	122	8.1141027E 72	4.3417099E 75
66	1.6394365E 59	1.6215523E 62	123	2.2733383E 75	1.8548992E 73
67	1.2120897E 62	1.8033066E 60	124	9.6367806E 72	5.0732986E 75
68	1.3313474E 60	1.2780935E 63	125	2.6164119E 75	2.1009426E 73
69	9.4489459E 62	1.3656242E 61	126	1.0750917E 73	5.5699987E 75
70	9.9754578E 60	9.3028268E 63	127	2.8286896E 75	2.2359112E 73
71	6.8016443E 63	9.5571711E 61	128	1.1266896E 73	5.7461170E 75
72	6.9065492E 61	6.2619379E 64	129	2.8728837E 75	2.2359112E 73
73	4.5273637E 64	6.1896136E 62	130	1.1092216E 73	5.5699987E 75
74	4.4246065E 62	3.9032204E 65	131	2.7409913E 75	2.1009426E 73
75	2.7903256E 65	3.7144373E 63	132	1.0258508E 73	5.0732986E 75
76	2.6262232E 63	2.2557875E 66	133	2.4566545E 75	1.8548992E 73
77	1.5943400E 66	2.0679487E 64	134	8.9122109E 72	4.3417099E 75
78	1.4459486E 64	1.2101476E 67	135	2.0682546E 75	1.5386743E 73
79	8.4552413E 66	1.0692791E 65	136	7.2726403E 72	3.4908673E 75
80	7.3930629E 64	6.0327392E 67	137	1.6355009E 75	1.1990957E 73
81	4.1663832E 67	5.1404604E 65	138	5.5739216E 72	2.6367071E 75
82	3.5139865E 65	2.7974760E 68	139	1.2146028E 75	8.7778660E 72
83	1.9094949E 68	2.2998366E 66	140	4.0117590E 72	1.8706259E 75
84	1.5541877E 66	1.2078259E 69	141	8.4701444E 74	6.0351167E 72
85	8.1472784E 68	9.5845933E 66	142	2.7110875E 72	1.2463365E 75
86	6.4022087E 66	4.8597229E 69	143	5.5455265E 74	3.8964040E 72
87	3.2390679E 69	3.7239257E 67	144	1.7198970E 72	7.7968667E 74
88	2.4583727E 67	1.8236655E 70	145	3.4079925E 74	2.3617265E 72
89	1.2006936E 70	1.3499703E 68	146	1.0240306E 72	4.5786796E 74
90	8.8064485E 67	6.3876106E 70	147	1.9654070E 74	1.3436166E 72
91	4.1551747E 70	4.5694607E 68	148	5.7206687E 71	2.5233669E 74
92	2.9451602E 68	2.0897832E 71	149	1.0633687E 74	7.1726186E 71
93	1.3427190E 71	1.4451800E 69	150	2.9979303E 71	1.3046992E 74
94	9.2017323E 68	6.3903092E 71	151	5.3958167E 73	3.5916927E 71
95	4.0548846E 71	4.2733958E 69	152	1.4731552E 71	6.3268140E 73
96	2.6875653E 69	1.8275443E 72	153	2.5669742E 73	1.6865018E 71

I	A(I)	B(I)	I	A(I)	B(I)
	N = 512				
0	0. E 0	0. E 0	55	4.3223409E 73	7.7631411E 71
1	0. E 0	1.0000000E 0	56	6.9292096E 71	3.2373267E 75
2	0. E 0	1.3081599E 5	57	2.9226415E 75	5.0602219E 73
3	1.7348496E 5	4.3434996E 4	58	4.5039861E 73	2.6317015E 77
4	4.3434996E 4	2.6243171E 9	59	1.8250912E 77	3.0594464E 75
5	3.3528056E 9	5.6155366E 8	60	2.7068929E 75	1.1603496E 79
6	5.5603741E 8	2.4247603E 13	61	1.0550648E 79	1.7115047E 77
7	2.7299719E 13	5.4300500E 12	62	1.5070653E 77	8.5621555E 80
8	3.3831574E 12	1.1064260E 17	63	5.6567621E 80	8.8931415E 78
9	1.2061283E 17	1.2125225E 16	64	7.7968600E 78	3.1661771E 82
10	1.1912066E 16	3.1165830E 20	65	2.6217532E 82	4.3001757E 80
11	3.3220881E 20	2.7833505E 19	66	3.7542546E 80	1.4662321E 84
12	2.7235576E 19	5.9380820E 23	67	1.5107026E 84	1.9360564E 82
13	6.2231545E 23	4.4694282E 22	68	1.6649976E 82	6.4830663E 85
14	4.3559463E 22	8.1403924E 26	69	5.6816856E 85	8.1638455E 83
15	8.4157004E 26	5.2888655E 25	70	7.0656396E 83	2.6401058E 87
16	5.1339182E 25	8.3949828E 29	71	2.3023970E 87	3.2165579E 85
17	6.5601557E 29	4.7932674E 28	72	2.7703395E 85	1.0006798E 89
18	4.6341159E 28	6.7357405E 32	73	8.7360355E 88	1.1874068E 87
19	6.8162194E 32	3.4271669E 31	74	1.0181099E 87	3.5995983E 90
20	3.3000060E 31	4.3169358E 35	75	3.1064299E 90	4.1138258E 88
21	4.3299943E 35	1.9792492E 34	76	3.5112146E 88	1.2087451E 92
22	1.6980691E 34	2.2572528E 38	77	1.0386673E 92	1.3393976E 90
23	2.2459145E 38	9.4106219E 36	78	1.1379649E 90	3.8170262E 93
24	8.9880696E 36	9.7981942E 40	79	3.2638987E 93	4.1036176E 91
25	9.6766392E 40	3.7428718E 39	80	3.4704420E 91	1.1349733E 95
26	3.5601143E 39	3.5824608E 43	81	9.6573148E 94	1.1845776E 93
27	3.5134216E 43	1.2619236E 42	82	9.9717369E 92	3.1616105E 96
28	1.1953767E 42	1.1169599E 46	83	2.6938569E 96	3.2256502E 94
29	1.0882244E 46	3.6480778E 44	84	2.7027420E 94	8.4161405E 97
30	3.4414484E 44	3.0013101E 48	85	7.0924470E 97	8.2950710E 95
31	2.9057153E 48	9.1321977E 46	86	6.9179596E 95	2.1046640E 99
32	8.5792714E 46	7.0144122E 50	87	1.7644097E 99	2.0166955E 97
33	6.7499365E 50	1.9966304E 49	88	1.6740130E 97	4.9769927E100
34	1.8679412E 49	1.4373917E 53	89	4.1318410E100	4.6400425E 98
35	1.3751038E 53	3.8416175E 51	90	3.8334725E 98	1.1143990E102
36	3.5790068E 51	2.6010630E 55	91	9.2502080E101	1.0113252E100
37	2.4741952E 55	6.5484000E 53	92	8.3157798E 99	2.3648631E103
38	6.0751756E 53	4.1827903E 57	93	1.9532343E103	2.0900248E101
39	3.9566782E 57	9.9485307E 55	94	1.7103914E101	4.7605651E104
40	9.1907324E 55	6.0114741E 59	95	3.9123345E104	4.0991167E102
41	5.6555682E 59	1.3543104E 58	96	3.3365563E102	9.0966297E105
42	1.2458597E 58	7.7008775E 61	97	7.4400644E105	7.6361695E103
43	7.2623633E 61	1.6600429E 60	98	6.1894693E103	1.6524166E107
44	1.5206252E 60	9.0419139E 63	99	1.3444270E107	1.3522713E105
45	8.4165497E 63	1.8402330E 62	100	1.0907969E105	2.8538739E108
46	1.6784938E 62	9.5466884E 65	101	2.3102703E108	2.2761888E106
47	8.8402038E 65	1.8523365E 64	102	1.8287604E106	4.6908576E109
48	1.6822978E 64	9.1696444E 67	103	3.7781896E109	3.6540795E107
49	8.4473891E 67	1.6992352E 66	104	2.9189814E107	7.3432590E110
50	1.5366130E 66	8.0405427E 69	105	5.8845956E110	5.5839152E108
51	7.3694765E 69	1.4253983E 68	106	4.4387761E108	1.0955904E112
52	1.2834153E 68	6.4573559E 71	107	8.7350392E111	8.1352285E109
53	5.8885010E 71	1.0967705E 70	108	6.4350903E109	[illegible]
54	9.8323764E 69	4.7638227E 73	109	1.2365722E113	1.1507217E111

I	A(I)	B(I)	I	A(I)	B(I)
154	6.7855346E 70	2.8763616E 73	206	1.5831564E 51	3.0169148E 53
155	1.1444646E 73	7.4226868E 70	207	9.7693319E 52	4.7522107E 50
156	2.9285536E 70	1.2254671E 73	208	9.0960398E 49	2.8547570E 52
157	4.7801238E 72	3.0610394E 70	209	5.3363991E 51	2.5711295E 49
158	1.1637613E 70	4.8908821E 72	210	4.7204331E 48	1.4673802E 51
159	1.8694278E 72	1.1821592E 70	211	2.6285642E 50	1.2545209E 48
160	4.4792754E 69	1.8275443E 72	212	2.2052125E 47	6.7903902E 49
161	6.8422739E 71	4.2733958E 69	213	1.1634454E 49	5.5008227E 46
162	1.5858304E 69	6.3903092E 71	214	9.2396629E 45	2.6185289E 48
163	2.3424900E 71	1.4451800E 69	215	4.6094732E 47	2.1592014E 45
164	5.2500681E 68	2.0897832E 71	216	3.4580961E 44	1.0451134E 47
165	7.4969498E 70	4.5694607E 68	217	1.6277309E 46	7.5546100E 43
166	1.6243005E 68	6.3876106E 70	218	1.1509280E 43	3.4464467E 45
167	2.2415311E 70	1.3499705E 68	219	5.0991499E 44	2.3451527E 42
168	4.6932571E 67	1.8236655E 70	220	3.3044785E 41	1.0057507E 44
169	6.2569223E 69	3.7239257E 67	221	1.4096090E 43	6.4248121E 40
170	1.2655528E 67	4.6597229E 69	222	8.7839228E 39	2.5629479E 42
171	1.6293433E 69	9.5845933E 66	223	3.4190157E 41	1.5443636E 39
172	3.1823844E 66	1.2078258E 69	224	1.9907613E 38	3.6017053E 40
173	3.9550934E 68	2.2998386E 66	225	7.2276102E 39	3.2358096E 37
174	7.4565080E 65	2.7974760E 68	226	3.9183633E 36	1.1318174E 39
175	8.9417905E 67	5.1404604E 65	227	1.5218117E 38	5.8658590E 35
176	1.6264737E 65	6.0327392E 67	228	6.6449186E 34	1.9025451E 37
177	1.6811376E 67	1.0692791E 65	229	2.0737104E 36	9.1225727E 33
178	3.2997267E 64	1.2101476E 67	230	9.6214638E 32	2.7308223E 35
179	3.6789292E 66	2.[illegible]679487E 64	231	2.7637921E 34	1.2053541E 32
180	6.2200022E 63	[illegible]557875E 66	232	1.1771036E 31	3.3121260E 33
181	6.6614869E 65	3.7144373E 63	233	3.0941432E 32	1.3378949E 30
182	1.0882140E 63	3.9032204E 65	234	1.2020147E 29	3.3533136E 31
183	1.1256150E 65	6.1896136E 62	235	2.8714627E 30	1.2310869E 28
184	1.7650070E 62	6.2619379E 64	236	1.0098760E 27	2.7934197E 29
185	1.7569140E 64	9.5571711E 61	237	2.1744870E 28	9.2443776E 25
186	2.6506216E 61	9.3028266E 63	238	6.8610615E 24	1.8818911E 27
187	2.5374470E 63	1.3656242E 61	239	1.3183847E 26	5.5581331E 23
188	3.6807840E 60	1.2780935E 63	240	3.6909477E 22	1.0039377E 25
189	3.3863419E 62	1.8033066E 60	241	6.2515361E 23	2.6137832E 21
190	4.7195902E 59	1.6215523E 62	242	1.5315136E 20	4.1312895E 22
191	4.1697491E 61	2.1973601E 59	243	2.2509131E 21	9.3339884E 18
192	5.5792351E 58	1.8969402E 61	244	4.7399160E 17	1.2681218E 20
193	4.7298238E 60	2.4668118E 58	245	5.9220562E 18	2.4357682E 16
194	6.0706685E 57	2.0427494E 60	246	1.0466188E 15	2.7773698E 17
195	4.9339511E 59	2.5470176E 57	247	1.0808106E 16	4.4095737E 13
196	6.0690656E 56	2.0213703E 59	248	1.5502510E 12	4.0806335E 14
197	4.7245722E 58	2.4142973E 56	249	1.2703377E 13	5.1413609E 10
198	5.5642010E 55	1.8345001E 58	250	1.4055068E 9	3.6709350E 11
199	4.1446587E 57	2.0967782E 55	251	8.5706940E 9	3.4412297E 7
200	4.6686078E 54	1.5238335E 57	252	6.8008495E 5	1.7410174E 8
201	3.3238908E 56	1.6649046E 54	253	2.7096717E 6	1.0794999E 4
202	3.5769435E 53	1.1559547E 56	254	0. E 0	3.2639999E 4
203	2.4312765E 55	1.2058642E 53	255	2.5500000E 2	1.0000000E 0
204	2.4965158E 52	7.9888502E 54	256	1.0000000E 0	0. E 0
205	1.6179547E 54	7.9468241E 51			

I	A(I)	B(I)
224	7.3763683E147	8.6156005E150
225	4.8488884E150	2.1581146E148
226	1.2097244E148	1.4004543E151
227	7.8267621E150	3.4529331E148
228	1.9220426E148	2.2055614E151
229	1.2239619E151	5.3527996E148
230	2.9580763E148	3.3655852E151
231	1.8544864E151	8.0403827E148
232	4.4127879E148	4.9764087E151
233	2.7225249E151	1.1702998E149
234	6.3772195E148	7.1302784E151
235	3.6728723E151	1.6506782E149
236	8.9304268E148	9.9003644E151
237	5.3385695E151	2.2562692E149
238	1.2118632E149	1.3321945E152
239	7.1313067E151	2.9888168E149
240	1.5936464E149	1.7372875E152
241	9.2315879E151	3.8370950E149
242	2.0309623E149	2.1957226E152
243	1.1581430E152	4.7743466E149
244	2.5083968E149	2.6896600E152
245	1.4081163E152	5.7576479E149
246	3.0025234E149	3.1933165E152
247	1.6592654E152	6.7298586E149
248	3.4832274E149	3.6746934E152
249	1.8949722E152	7.6243829E149
250	3.9164310E149	4.0986556E152
251	2.0975223E152	8.3723621E149
252	4.2679421E149	4.4310734E152
253	2.2502588E152	8.9112940E149
254	4.5078616E149	4.6433115E152
255	2.3398298E152	9.1936159E149
256	4.6147639E149	4.7162899E152
257	2.3581091E152	9.1936159E149
258	4.5786516E149	4.6433115E152
259	2.3034134E152	8.9112940E149
260	4.4034323E149	4.4310734E152
261	2.1607553E152	8.3723621E149
262	4.1044197E149	4.0986556E152
263	2.0010862E152	7.6243829E149
264	3.7079517E149	3.6746935E152
265	1.7796931E152	6.7298586E149
266	3.2466310E149	3.1933165E152
267	1.5340424E152	5.7576479E149
268	2.7551244E149	2.6896600E152
269	1.2815487E152	4.7743466E149
270	2.2659497E149	2.1957226E152
271	1.0375971E152	3.8370950E149
272	1.8061326E149	1.7372875E152
273	8.1415503E151	2.9888168E149
274	1.3951703E149	1.3321945E152
275	6.1909484E151	2.2562692E149
276	1.0444058E149	9.9003644E151
277	4.5620950E151	1.6506782E149
278	7.5763550E148	7.1302784E151
279	3.2577084E151	1.1702998E149
280	5.3257785E148	4.9764087E151
281	2.2541500E151	8.0403827E148
282	3.6275946E148	3.3655852E151
283	1.5113194E151	5.3527996E148
284	2.3941233E148	2.2055614E151
285	9.6177290E150	3.4529331E148
286	1.5506901E148	1.4004543E151
287	6.1790799E150	2.1581146E148
288	9.4839021E147	8.6156005E150
289	3.7676406E150	1.3008109E148
290	5.6917999E147	5.1350254E150
291	2.2254666E150	7.6662268E147
292	3.3090547E147	2.9649137E150
293	1.2733566E150	4.3365641E147
294	1.8634607E147	1.6583029E150
295	7.0270881E149	2.3961556E147
296	1.0164058E147	8.9039287E149
297	3.7660471E149	1.2766236E147
298	5.3692201E146	4.7139390E149
299	1.9691785E149	6.6024801E146
300	2.7467374E146	2.3954475E149
301	9.9128479E148	3.3016616E146
302	1.3300340E146	1.1707769E149
303	4.8316320E148	1.5967751E146
304	6.9262504E145	5.6166971E148
305	2.2603484E148	7.4958971E145
306	3.0305678E145	2.5911552E148
307	1.0418372E148	3.4025255E145
308	1.5623393E145	1.1372453E148
309	4.6078038E147	1.4951206E145
310	5.9279196E144	5.0030110E147
311	1.9724779E147	6.3591866E144
312	2.4964775E144	2.0934563E147
313	8.1717356E146	2.6177531E144
314	1.0174470E144	8.4776020E146
315	3.2760442E146	1.0428138E144
316	4.0123891E143	3.3220549E146
317	1.2707663E146	4.0195991E143
318	1.5309019E143	1.2595375E146
319	4.7687777E145	1.4989988E143
320	5.6505229E142	4.6198674E145
321	1.7310773E145	5.4076034E142
322	2.0172694E142	1.6390916E145
323	6.0776281E144	1.8868328E142
324	6.9650064E141	5.6243339E144
325	2.0634090E144	6.3666513E141
326	2.3253929E141	1.8662490E144
327	6.7739713E143	2.0773702E141
328	7.5061232E140	5.9674230E143
329	2.1498250E143	6.5529029E140
330	2.3421509E140	1.6569138E143
331	6.5948951E142	1.9980660E140
332	7.0635481E139	5.5604162E142
333	1.9551720E142	5.8882070E139
334	2.0565723E139	1.6125400E142
335	5.6009220E141	1.6767368E139
336	5.7965305E138	4.5135651E141
337	1.5500784E141	4.6129803E138

I	A(I)	B(I)	I	A(I)	B(I)
110	8.9000162E110	2.1160444E114	167	3.1153682E138	1.8052499E136
111	1.6705513E114	1.5002759E112	168	1.2506657E136	1.9573406E139
112	1.1750207E112	2.7448485E115	169	1.3177419E139	7.7968593E136
113	2.1550340E115	1.9014241E113	170	5.2232732E136	6.0366766E139
114	1.4817739E113	3.4006971E116	171	5.3800325E139	3.1462559E137
115	2.6561961E116	2.3032033E114	172	2.0934542E137	3.1874295E140
116	1.7858822E114	4.0279649E117	173	2.1206208E140	1.2256928E138
117	3.1296651E117	2.6679200E115	174	8.1167513E137	1.2204606E141
118	2.0562647E115	4.5630263E118	175	6.0715441E140	4.6129603E138
119	3.5276772E118	2.9569124E116	176	3.0362778E138	4.5133652E141
120	2.2696612E116	4.9484605E119	177	2.9672014E141	1.6767365E139
121	3.6052138E119	3.1372594E117	178	1.0970834E139	1.6125400E142
122	2.3958361E117	5.1379316E120	179	1.0537014E142	5.8862070E139
123	3.9302219E120	3.1680625E118	180	3.8296345E139	5.5664162E142
124	2.4221803E118	5.1106454E121	181	3.6163267E142	1.9960866E140
125	3.8887965E121	3.1043913E119	182	1.2917317E140	1.8569138E143
126	2.3464831E119	4.8723431E122	183	1.1987076E143	6.5529029E140
127	3.6878931E122	2.8980145E120	184	4.2107521E140	5.9873230E143
128	2.1791710E120	4.4542254E123	185	3.8413901E143	2.0773702E141
129	3.3535413E123	2.5947336E121	186	1.3267579E141	1.8662490E144
130	1.9409825E121	3.9063318E124	187	1.1899919E144	6.3666513E141
131	2.9253742E124	2.2291552E122	188	4.0414582E141	5.6243339E144
132	1.6568049E122	3.2876518E125	189	3.5640669E144	1.8868326E142
133	2.4490349E125	1.8383291E123	190	1.1903261E142	1.6390916E145
134	1.3607945E123	2.6569206E126	191	1.0322079E145	5.4076034E142
135	1.9664358E126	1.4556470E124	192	3.3903136E142	4.6196674E145
136	1.0719810E124	2.0622392E127	193	2.8911060E145	1.4969966E143
137	1.5196055E127	1.1076062E125	194	9.5394650E142	1.2595375E146
138	8.1123506E124	1.5380076E128	195	7.6324669E145	4.0195991E143
139	1.1271676E128	8.0983098E125	196	2.4886970E143	3.3220550E146
140	5.8997452E125	1.1025443E129	197	2.0527341E146	1.0428138E144
141	8.0362451E128	5.6924472E126	198	6.4157487E143	8.4776020E146
142	4.1246004E126	7.5998576E129	199	5.2049828E146	2.6177331E144
143	5.5090704E129	3.6481345E127	200	1.6003061E144	2.0734563E147
144	2.7733624E127	5.0386915E130	201	1.2770662E147	6.3591666E144
145	3.6325543E130	2.5026142E128	202	3.8627089E144	5.0030110E147
146	1.7938661E128	3.2146079E131	203	3.0322621E147	1.4951206E145
147	2.3046112E131	1.5662852E129	204	9.0232662E144	1.1572453E148
148	1.1165900E129	1.9730697E132	205	6.9683320E147	3.4025255E145
149	1.4072564E132	9.4366342E129	206	2.0401860E145	2.5911552E148
150	6.6904260E129	1.1669530E133	207	1.5500547E148	7.4958971E145
151	8.2731773E132	5.4747475E130	208	4.4653292E145	5.6166971E148
152	3.8601246E130	6.6442903E133	209	3.3378490E148	1.5967751E146
153	4.6840758E133	3.0594176E131	210	9.4615017E145	1.1707769E149
154	2.1451775E131	3.6444617E134	211	6.9587456E148	3.3016818E146
155	2.5547748E134	1.6472662E132	212	1.9410278E146	2.3954479E149
156	1.1465821E132	1.9263194E135	213	1.4046918E149	6.6024861E146
157	1.3426996E135	8.5478672E132	214	3.8557486E146	4.7139590E149
158	5.9267437E132	9.8140870E135	215	2.7457192E149	1.2766236E147
159	6.8017228E135	4.2759749E133	216	7.4170155E146	8.9839267E149
160	2.9480842E133	4.8207072E136	217	5.1974870E149	2.3961556E147
161	3.3218902E136	2.0625602E134	218	1.3817497E147	1.6583029E150
162	1.4139816E134	2.2839979E137	219	9.5285813E149	4.3305649E147
163	1.5645391E137	9.5957557E134	220	2.4931234E147	2.9649157E150
164	6.5408565E134	1.0434740E138	221	1.6919751E150	7.8[illegible]E147
165	7.1076717E137	4.3068132E135	222	4.3371709E147	5.1350255E150
166	2.9188753E135	4.6004286E138	223	2.9101949E150	1.30681[illegible]E148

I	A(I)	B(I)	I	A(I)	B(I)
338	1.5767022E138	1.2204606E141	395	1.0503246E110	2.6079260E115
339	4.1436651E140	1.2250926E138	396	6.0966325E114	4.0279649E117
340	4.1421769E137	3.1874295E140	397	9.1131725E110	2.5052055E114
341	1.0697324E140	3.1462539E137	398	5.1732107E113	5.4000971E110
342	1.0507997E137	6.0360766E139	399	7.5012435E115	1.9014241E113
343	2.6664426E139	7.7960393E136	400	4.1965027E112	2.7440405E115
344	2.5735060E136	1.9573466E139	401	5.9956563E114	1.5002759E112
345	6.4160590E138	1.8652499E136	402	3.2525513E111	2.1106444E114
346	6.0639206E135	4.6004268E138	403	4.5414152E113	1.1507217E111
347	1.4900143E138	4.5060132E135	404	2.4072005E110	1.5509125E113
348	1.3679376E135	1.0434740E138	405	3.2053596E112	6.1532255E109
349	3.3388917E137	9.5957557E134	406	1.7001349E109	1.0955904E112
350	3.0548987E134	2.2835979E137	407	2.2649209E111	5.5039152E108
351	7.2177925E136	2.0625602E134	408	1.1451368E108	7.3432590E110
352	6.4857852E133	4.8207072E136	409	1.4844155E110	3.6540795E107
353	1.5048506E136	4.2759749E133	410	7.5509602E106	4.6906576E109
354	1.3278906E133	9.8140670E135	411	9.3312742E108	2.2761668E106
355	3.0252602E135	8.5478672E132	412	4.4940835E105	2.6556759E108
356	2.6211233E132	1.9263194E135	413	5.5656792E107	1.5522713E105
357	5.6627557E134	1.0472662E132	414	2.6147434E104	1.6524166E107
358	4.9866413E131	3.6444616E134	415	3.1560662E106	7.6361695E103
359	1.0949547E134	3.0594176E131	416	1.4466999E103	9.0986296E105
360	9.1424003E130	6.6442903E133	417	1.7034151E105	4.0991167E102
361	1.9702781E133	5.4747475E130	418	7.6057831E101	4.7605651E104
362	1.6146226E130	1.1669530E133	419	8.7267926E103	2.0900248E101
363	3.4148633E132	9.4366342E129	420	3.7963343E100	2.3646631E103
364	2.7462060E129	1.9738897E132	421	4.2428452E102	1.0113252E100
365	5.6991061E131	1.5662852E129	422	1.7974727E 99	1.1143990E102
366	4.4969516E128	3.2146078E131	423	1.9558775E101	4.6400425E 98
367	9.1558043E130	2.5026142E128	424	8.0656990E 97	4.9769927E100
368	7.0874819E127	5.0308913E130	425	8.5408930E 99	2.0166933E 97
369	1.4154696E130	3.8481345E127	426	3.4266032E 96	2.1046040E 99
370	1.0747719E127	7.5996574E129	427	3.5295364E 98	8.2950710E 95
371	2.1052157E129	5.6924472E126	428	1.3771113E 95	8.4161403E 97
372	1.5676465E126	1.1025443E129	429	1.3789212E 97	3.2256502E 94
373	3.0110684E128	8.0983098E125	430	5.2290814E 93	3.1616162E 96
374	2.1985645E125	1.5380075E128	431	5.0874598E 95	1.1645776E 93
375	4.1402624E127	1.1076062E125	432	1.8740388E 92	1.1349755E 95
376	2.9637122E124	2.0622392E127	433	1.7705586E 94	4.1056176E 91
377	5.4709417E126	1.4558470E124	434	6.3317537E 90	5.6170262E 93
378	3.8386589E123	2.6569206E126	435	5.6056356E 92	1.3595970E 90
379	6.9448267E125	1.8365291E123	436	2.0143283E 89	1.2067451E 92
380	4.7753473E122	5.2676517E125	437	1.7913224E 91	4.1150256E 88
381	8.4656081E124	2.2291552E122	438	6.0261105E 87	3.5995963E 90
382	5.7035023E121	3.9063317E124	439	5.1940554E 89	1.1674068E 87
383	9.9055479E123	2.5947338E121	440	1.6929652E 86	1.0006796E 89
384	6.5375131E120	4.4542253E123	441	1.4133191E 88	3.2163579E 85
385	1.1120960E123	2.8980145E120	442	4.4601838E 84	2.6401058E 87
386	7.1884325E119	4.8723431E122	443	3.6035563E 86	8.1636455E 83
387	1.1974646E122	3.1043913E119	444	1.1002057E 83	6.4650665E 85
388	7.5790808E118	5.1106454E121	445	8.5960360E 84	1.9366964E 82
389	1.2360781E121	3.1880625E118	446	2.5369061E 81	1.4602321E 84
390	7.6588206E117	5.1379316E120	447	1.9152157E 83	4.3001757E 80
391	1.2226184E120	3.1372594E117	448	5.4592074E 79	3.1661770E 82
392	7.4142263E116	4.9484665E119	449	3.9785167E 81	6.0931413E 78
393	1.1582144E119	2.9567124E116	450	1.0942731E 78	6.3621531E 80
394	6.6725111E115	4.5636263E118	451	7.6911322E 79	1.7115697E 77

I	A(I)	B(I)
452	2.0391926E 76	1.1803496E 79
453	1.3808679E 78	3.0594464E 75
454	3.5255339E 74	2.0317014E 77
455	2.2975971E 76	5.0682219E 73
456	5.6423564E 72	3.2373267E 75
457	3.5347308E 74	7.7631411E 71
458	8.3393117E 70	4.7638227E 73
459	5.0156471E 72	1.0967705E 70
460	1.1353289E 69	6.4375559E 71
461	6.5468377E 70	1.4253963E 68
462	1.4198304E 67	6.0405427E 69
463	7.8383525E 68	1.6992352E 66
464	1.6262211E 65	9.1696[illegible]44E 67
465	8.5814191E 66	1.8523365E 64
466	1.7003871E 63	9.5466883E 65
467	6.5619352E 64	1.6402330E 62
468	1.6173922E 61	9.0419139E 63
469	7.7565827E 62	1.6600429E 60
470	1.3941763E 59	7.7608772E 61
471	6.3549705E 60	1.3543104E 58
472	1.0845063E 57	6.0114741E 59
473	4.6880310E 58	9.9485307E 55
474	7.5779822E 54	4.1827903E 57
475	3.0988127E 56	6.5484000E 53
476	4.7322423E 52	2.6010630E 55
477	1.8255560E 54	3.8416175E 51
478	2.6261057E 50	1.4373917E 53
479	9.5277872E 51	1.9966304E 49
480	1.2868906E 48	7.0144122E 50
481	4.3759811E 49	9.1321977E 46
482	5.5292603E 45	3.0013101E 48
483	1.7553453E 47	3.6480778E 44
484	2.0662941E 43	1.1169599E 40
485	6.0970067E 44	1.2017250E 42
486	[illegible] 40	3.3024007E 43
487	1.8130409E 42	3.7420716E 39
488	1.8273742E 38	9.7701942E 40
489	4.5043361E 39	9.4108219E 36
490	4.2275176E 35	2.2572526E 38
491	9.6809640E 36	1.9792492E 34
492	8.1180149E 32	4.3169358E 35
493	1.6831302E 34	3.4271869E 31
494	1.2718076E 30	6.7337403E 32
495	2.3635588E 31	4.7932674E 28
496	1.5915145E 27	8.3949828E 29
497	2.61[illegible]531E 28	5.2888655E 25
498	1.5494723E 24	8.1403923E 26
499	2.2216460E 25	4.4694282E 22
500	1.1348157E 21	5.9366620E 23
501	1.3890743E 22	2.7633365E 19
502	5.9798674E 17	3.1105830E 20
503	6.0753773E 18	1.2125223E 16
504	2.1313891E 14	1.1064267E 17
505	1.7254357E 15	3.4300500E 12
506	4.6892488E 10	[illegible]4247608E 13
507	2.8360110E 11	5.6155360E 8
508	5.3162444E 6	2.8243171E 9
509	2.2021798E 7	4.3434998E 4
510	0. E 0	1.3081599E 5
511	5.1099998E 2	1.0000000E 0
512	1.0000000E 0	0. E 0